# A TEXT BOOK OF
# PHYSICS

## FOR STUDENTS OF SCIENCE AND ENGINEERING

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*PART I*
*DYNAMICS*

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# PREFACE

THE preparation of this volume was undertaken to meet a demand that has been growing for some years past for a text-book of Physical Science which should connect more intimately than has hitherto been usual the scientific aspects of Physics with its modern practical applications. The reader must be left to judge how far the authors have succeeded in thus combining the outlooks of the man of science and the engineer.

The contents have been selected to meet the requirements of various classes of students: those preparing for Intermediate and other examinations of London and other Universities; and those entering for appointments in the Army, Navy, and Civil Service, or offering themselves for examination in Electrical Engineering (Grade I) by the City and Guilds of London Institute.

The book has been arranged in parts, in accordance with the divisions of the subject found convenient in most schools and colleges. Part I, Dynamics, comprises the sections of Mechanics and Applied Mathematics usually studied, and includes sections on motion, statics, and the properties of fluids. Part II, Heat; Part III, Light; Part IV, Sound; and Part V, Magnetism and Electricity; deal respectively with the principles of these subjects and their applications.

Complete courses of laboratory work have been provided in each Part. Many physical laboratories are equipped with apparatus differing in some respects from the instruments here described, nevertheless the guidance given will enable intelligent use to be made of other forms of apparatus designed for the same or similar purposes.

Attention is directed to the experimental treatment of dynamical principles, because its neglect, which is unfortunately common, makes it difficult for a student to secure a thorough and systematic knowledge of physical science. The complete course of experimental work has been devised to meet both the requirements of the physicist and of the engineer; in cases where the methods of treatment adopted by these differ radically, the teacher or student may choose the experiment which best suits his special needs.

In Part V, the treatment of the Dynamo, Telegraph, and so on, is that which follows naturally and logically from the earlier theoretical principles explained; technical considerations of design and construction have been omitted as unsuitable in a text-book of Physics.

A large number of worked-out examples have been included to assist the student to understand the text and to solve the exercises at the ends of the chapters. Many of these exercises have been taken, with the permission of the authorities to whom grateful acknowledgments are made, from examination papers, the source being given in each case. Questions marked L.U. are from examination papers of the London University and those with C.G. from papers of the City and Guilds of London Institute.

Answers have been supplied in the case of numerical exercises, but it is too much to hope that these are entirely free from errors. The authors will welcome any corrections which readers may send to them.

The authors are glad of this opportunity to express their indebtedness to Prof. Sir Richard Gregory and Mr. A. T. Simmons for constant assistance and invaluable hints while the book was in preparation and passing through the press.

J. DUNCAN
1918 S. G. STARLING

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# *PART I*—DYNAMICS

## CHAPTER I

### INTRODUCTORY

**Preliminary definitions.**—**Dynamics** is that branch of physical science which investigates the behaviour of **matter** under the action of **force.**

It must suffice here to explain what is meant by matter by reference to some of its properties, of which the most obvious are : (i) it always occupies space, (ii) it always possesses weight when in the neighbourhood of the earth. A **body** is any definite portion of matter.

**Force** is push or pull exerted on a body ; and may alter the state of motion by causing the speed of the body to increase or decrease continuously, or by producing a continuous change in the direction of motion. Our earliest appreciation of force comes usually by reason of the muscular effort which has to be exerted in sustaining the weight of a body.

**Statics** is that branch of the subject dealing with cases in which the forces do not produce any change in the motion of the body to which they are applied. **Kinetics** includes all problems in which change of motion occurs as a consequence of the application of force to the body. Another subdivision called **Kinematics** deals with the mere geometry of motion without reference to the applied force.

In another nomenclature in common use, the name **mechanics** is given to the entire subject, and **dynamics** to that branch in which the applied forces produce changes in the motion of the body ; in this nomenclature **statics** and **kinematics** have the signification defined above.

**Fundamental units.**—The fundamental units—to which are referred all measurements in any scientific system—are those of **length, mass** and **time.** Other units, obtained from these, are called **derived** units.

The metric unit of length is the **metre,** and may be defined as the distance, at a temperature of 0° C., between the ends of a standard bar preserved in Paris. Other practical units are the *centimetre* (0·01 metre, written one cm.), the *millimetre* (0·001 metre, written one mm.) and the *kilometre* (1000 metres).

The British unit of length is the **foot,** which is one-third of the standard *yard.* The latter may be defined as the distance between two marks on a standard bar preserved in London when the temperature of the bar is 62° F. The *inch* (one-twelfth of a foot) and the *mile* (5280 feet) are other practical units. One inch equals 2·539 cm., and

one metre equals 39·37 inches. For convenience in showing dimensions in drawings, lengths such as 3 feet 5 inches are written 3′ 5″.

Units used in measuring areas are produced by taking squares having sides equal to any of the units of length mentioned above, and are described as the square centimetre, the square inch, etc.

In measuring volumes, units are obtained by taking cubes having edges equal to any of the units of length, and are described as the cubic centimetre (written one c.c.), the cubic inch, etc. Other units of volume are the **litre** (1000 c.c., equal to 1·762 pint), the **gallon** (0·1605 cubic foot, or 8 pints, or 4·541 litres) and the **pint**.

**Mass** means quantity of matter. The metric unit of mass was intended to be the quantity of matter contained in a cubic centimetre of pure water at a temperature of 4 degrees Centigrade, but is actually one-thousandth of the mass of a piece of platinum preserved in Paris; this unit is called one **gram**. The *kilogram* (1000 grams) is another unit in common use. The British unit of mass is called the **pound avoirdupois** and is the quantity of matter contained in a standard piece of platinum preserved in London. The *ton* (2240 pounds) is also used often. One gallon of fresh water has a mass of 10 pounds. One pound equals 453·6 grams.

The unit of time employed in all scientific systems is the **second**, or 1/86400 of the mean solar day, which is the average time elapsing between two successive passages of the sun across the meridian of any one place on the surface of the earth.

It will be noted that the units of length, mass and time, on being once stated for any system of scientific measurement, remain invariable. Owing to the three metric units in common use being the centimetre, the gram, and the second, the name **C.G.S. system** is used more frequently than the term metric system.

For some purposes the metre, kilogram and second are taken as the fundamental units. This, the **M.K.S.** system, has not much to recommend it for dynamical purposes, but it has the advantage of making the practical units in electricity, namely, the volt, ampere and ohm (see Chap. LXVI) the absolute units without introducing powers of 10.

**Density.**—The **density** of a given material means the mass contained in unit volume of the material. In the C.G.S. system it is customary to measure density in grams per cubic centimetre; in the British system densities are stated usually in pounds per cubic foot or per cubic inch.

Let $\mathsf{V}$ = the volume of a body,

$d$ = the density of the material,

$m$ = the mass of the body.

Then $$m = Vd,$$

or $$d = \frac{m}{V}.$$

AVERAGE DENSITIES OF COMMON MATERIALS *

| Material | Density: Grams per c.c. | Density: Pounds per cubic ft. | Material | Density: Grams per c.c. | Density: Pounds per cubic ft. |
|---|---|---|---|---|---|
| Aluminium | 2·65 | 164 | Cork | 0·24 | 15 |
| Brass | 8·6 | 535 | Deal | 0·6 | 37·5 |
| Copper | 8·93 | 555 | Ebony | 1·2 | 75 |
| Gold | 19·32 | 1200 | Oak | 0·8 | 50 |
| Gunmetal | 8·2 | 510 | Pitch pine | 0·65 | 41 |
| Iron, Cast | 7·2 | 450 | | | |
| ,, Wrought | 7·8 | 480 | Granite | 2·7 | 168 |
| Lead | 11·37 | 710 | Marble | 2·6 | 162 |
| Platinum | 21·5 | 1340 | Sandstone | 2·25 | 140 |
| Silver | 10·5 | 655 | | | |
| Steel | 7·8 | 480 | Glass, Flint | 3·7 | 230 |
| Tin | 7·29 | 455 | ,, Crown | 2·5 | 156 |
| Fresh water | 1·0 | 62·3 | Indiarubber | 0·95 | 59 |
| Sea water | 1·03 | 64 | Leather | 0·9 | 56 |

**Dimensions of a quantity.**—The dimensions of any physical quantity may be stated in terms of the fundamental units. Using the symbols $l$, $m$ and $t$ to denote length, mass and time respectively, the dimensions of area, volume and density will be $l^2$, $l^3$ and $m/l^3$ or $ml^{-3}$ respectively.

EXAMPLE.—Suppose that in obtaining a certain result, the final calculation takes the form

$$\frac{12\ (\text{grams}) \times 3\ (\text{cm.}) \times 3\ (\text{cm.})}{6\ \text{cm.} \times 2\ (\text{sec.}) \times 2\ (\text{sec.})}.$$

The numerical result is 4·5. To obtain the dimensions, cancel corresponding bracketed quantities in the numerator and denominator, giving :

$$\frac{\text{grams} \times \text{cm.}}{\text{sec.} \times \text{sec.}}, \text{ or, gm. cm. sec.}^{-2},$$

that is, $$\frac{ml}{t^2}, \text{ or, } mlt^{-2}.$$

It will be seen later that this result indicates a force.

* For fuller Tables of densities, see *Physical and Chemical Constants*, by Kaye and Laby. (Longmans.)

**Gravitation.**—There is a universal tendency of every body to move towards every other body; every particle of matter attracts every other particle towards itself with a force in the direction of the line joining the particles. The forces of attraction between bodies of small or moderate size are very small, but, when one or both bodies is large, the forces become evident without the necessity for employing delicate means for their detection. What we call the **weight** of a body is really the attractive force which the earth exerts on the body, tending to cause the body to approach the earth's centre. The term **gravitation** is applied to this universal attraction.

Gravitational effect takes place over unlimited distances; thus the force of attraction which the sun exerts on the earth causes the earth to describe an orbit round the sun. The force of attraction between two small bodies is proportional to the product of their masses, and is inversely proportional to the square of the distance between them. Expressed algebraically:

$$\mathsf{F} \propto \frac{m_1 m_2}{d^2},$$

where $\mathsf{F}$ is the force, $m_1$ and $m_2$ are the masses of the bodies and $d$ is the distance between them. We may also write

$$\mathsf{F} = k\frac{m_1 m_2}{d^2},$$

in which $k$ is a numerical constant called the **constant of gravitation.** The value of $k$ is about $6{\cdot}65 \times 10^{-8}$, expressed in C.G.S. units,* hence, expressed in dynes (p. 6),

$$\mathsf{F} = 6{\cdot}65 \times 10^{-8}\frac{m_1 m_2}{d^2} \text{ dynes.}$$

The same law holds for uniform spherical masses, where $d$ is the distance between their centres, a fact first established mathematically by Sir Isaac Newton.

**Weight.**—The **weight** of any given body varies somewhat, depending on the latitude of the place where the observation is made, and on the distance of the body above or below the surface of the earth. Weight is always directed vertically downwards.

**Equal masses situated at the same place possess equal weights.** It follows from this fact that a common **balance** (Fig. 1) may be used for obtaining a body having a mass equal to any standard mass. A standard mass may be placed in the scale pan A, and material may be added

* C. V. Boys, *Proc. R. Soc.*, London, 1894. The mean density of the earth has been determined and is given by Boys to be 5·527, or approximately $5\frac{1}{2}$ times that of water.

to, or taken away from, the scale pan B until the weights acting on A and B are equal, as will be evidenced by the balance beam CD becoming horizontal, or vibrating so that it describes small equal angles above and below the horizontal. The mass in A will then be equal to that in B. The use of such a balance is facilitated by a vertical pointer fixed to the beam and vibrating over a graduated scale. Assuming that the balance is properly adjusted, the weights are equal when the pointer swings through equal angles on each side of the middle division.

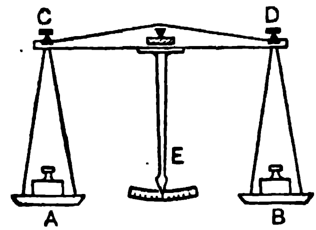

FIG. 1.—A common balance.

Standard masses ranging from 1 kilogram to 0·01 gram, and from 1 pound to 0·001 pound are provided in most laboratories. These are generally called **sets of weights**; the operation involved in using them is described as **weighing**.

**Units of force.**—For many practical purposes the weight of the unit of mass is employed as a unit of force. As has been explained, this weight is variable, hence the unit is not strictly scientific. The name **gravitational unit of force** is given to any force unit based on weight. The C.G.S. and British gravitational force units are respectively the weight of one gram mass, written **one gram weight**, and the weight of the pound mass, written **one lb. weight**. The kilogram weight and the ton weight are other convenient gravitational units of force.

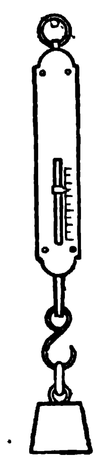
FIG. 2.—Spring balance.

A common balance cannot be used for showing the variation in weight of a body. **Spring balances** (Fig. 2), if of sufficiently delicate construction, might be employed for this purpose. It is known that a helical spring extends by amounts proportional to the pull applied, and in spring balances advantage is taken of this property. The body to be weighed is hung from the spring, and the extension is indicated by a pointer moving over a scale. For convenience, the scale is graduated in gram-weight or lb.-weight units, so as to enable the weight to be read direct. Such a balance will give correct readings of weight at the place where the scale was graduated, but, if used in a different latitude, will give a different reading when the same body is suspended from the balance. It may be noted that the variation in weight all over the earth is very small.

**Absolute units of force** are based on the fundamental units of length, mass and time, and are therefore invariable. The absolute unit of force in any system is the force which, if applied during one second to a body of unit mass, initially at rest, will give to the body a velocity of one unit of length per second. The C.G.S. absolute force unit is called the **dyne**; one dyne applied to one gram mass during one second will produce a velocity of one centimetre per second. The British absolute force unit is the **poundal**, and, if applied to a one pound mass during one second, will produce a velocity of one foot per second. These units will be referred to later and explained more fully.

## Exercises on Chapter I

**1.** Given that 1 metre = 39·37 inches, obtain a factor for converting miles to kilometres; use the factor to convert 3 miles 15 chains to kilometres. 80 chains = 1 mile.

**2.** Convert 2·94 metres to feet and inches.

**3.** A thin circular sheet of iron has a diameter of 14 cm. Find its area, taking $\pi = \frac{22}{7}$. If the material weighs 0·1 kilogram per square metre, find the weight of the sheet.

**4.** Calculate the volume of a ball 9 inches in diameter. Find the mass in pounds if the material has a density of 450 pounds per cubic foot.

**5.** A masonry wall is trapezoidal in section, one face of the wall being vertical. Height of wall, 20 feet; thickness at top, 4 feet; thickness at base, 9 feet. The masonry weighs 150 lb. per cubic foot. Find the weight of a portion of the wall 1 foot in length.

**6.** Find the weight of a solid pyramid of lead, having a square base of 4 inches edge and a vertical height of 8 inches. Lead weighs 0·41 lb. per cubic inch.

**7.** A hollow conical vessel has an internal diameter of 6 inches at the top and is 9 inches deep inside. Calculate the weight of water which it can contain. Water weighs 0·036 lb. per cubic inch.

**8.** Calculate the diameter of a solid ball of cast iron so that the weight may be 90 lb. The material weighs 0·26 lb. per cubic inch.

**9.** Three small bodies, A, B and C, of masses 2, 3 and 4 grams respectively, are arranged at the corners of a triangle having sides AB = 8 cm., BC = 12 cm., CA = 10 cm. Compare the gravitational efforts which A exerts on B, A exerts on C, C exerts on B.

**10.** If the distance from the earth to the moon is 240,000 miles, and from the sun to the moon is ninety million miles, determine the ratio of the gravitational forces of the sun and earth upon the moon, having given that the mass of the sun is 330,000 times that of the earth.

**11.** Distinguish between mass and weight. How are the mass and weight of a body affected by (*a*) variations of latitude, (*b*) variations of altitude?

If a very delicate balance is required for a laboratory near the top of a high mountain, would you advise having the weights specially adjusted for that altitude? Give careful reasons for your answer.

**12.** What is meant by *weight?* Explain why a very delicate spring balance would show slight differences in the weight of a body at different places on the earth, though a common balance would give no indication of any differences. L.U.

**13.** State the Law of Gravitation, and give a brief account of the facts which led to its discovery.

A spherical mass of 20 kgm., situated at the earth's surface, is attracted by another spherical mass of 150 kgm., with a force equal to the weight of 0·25 milligram, when the centres of the masses are 30 cm. apart. Calculate, by the aid of this result, the approximate mass and mean density of the earth, assuming the earth's radius to be $6 \times 10^8$ cm. C.W.B., H.C.

# CHAPTER II

## SIMPLE MEASUREMENTS AND MEASURING APPLIANCES

**Introductory experiments.**—The experiments described in this chapter are intended to render the student familiar with the use of simple measuring appliances.

EXPT. 1.—**Scales.** Laboratory scales have generally one edge graduated in centimetres subdivided to millimetres, and the other edge graduated in inches subdivided to tenths. Reproduce a portion of one of these scales in the following manner : Take a strip of cardboard of suitable width and rule lines lengthwise on it, agreeing with those on the scale. Arrange the scale and the cardboard end to end on the bench, and fasten them to prevent slipping. Set a beam compass to a radius of about 40 cm. The compass should have a hard pencil with a sharp chisel point, or a drawing pen charged with Indian ink. Stand the needle leg of the compass successively on the marks of the scale, and mark the cardboard with corresponding lines, prolonging slightly every fifth line. Insert the numbers on the cardboard scale as in Fig. 3.

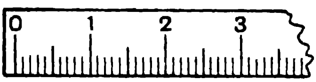


FIG. 3.—Centimetre scale.

EXPT. 2.—**Use of scales and calipers.** Several bodies of different shapes and materials are supplied. Make clear sketches of each. By means of a scale applied to the body, or by first fitting outside calipers A (Fig. 4), or inside calipers B, and then applying the calipers to the scale, measure all the dimensions of the body and insert them in the sketches. Fig. 5 shows suitable dimensioned sketches of a hollow cylinder.

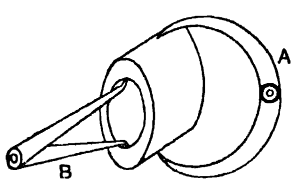


FIG. 4.—Use of calipers.

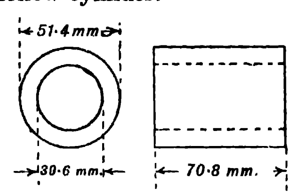


FIG. 5.—A hollow cylinder.

Calculate the volume of each body by application of the rules of mensuration, making use of the dimensions measured.

The student should practise the estimation by eye to one-tenth of a scale division.

**Verniers.**—Scales do not usually have divisions smaller than half a millimetre. Finer subdivision may be obtained by means of a **vernier**, an appliance which enables greater accuracy to be obtained than is possible by mere eye estimation.

In Fig. 6, A is a scale and B is a vernier ; B may slide along the edge of A. The divisions on the vernier from 0 to 10 have a total length of 9 scale divisions ; hence each vernier division is 0·9 of a scale division. If B is moved so that the mark 1 on the vernier is in the same straight line as the 0·1 mark on A, then the distance separating 0 on the vernier from 0 on the scale will be one-tenth of a scale division. If the mark 2 on the vernier is in line with the 0·2 mark on A, the distance separating

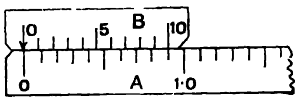

FIG. 6.

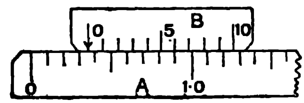

FIG. 7.

Forward-reading verniers.

the zero marks will be two-tenths of a scale division, and so on. The vernier thus enables readings to be taken to one-tenth of a scale division by simply noting which division on the vernier is in line with any particular mark on A. Fig. 7 shows a vernier and scale reading 0·36 scale divisions ; 0·3 is read from the scale, and the 6 from the vernier.

The appliance described above is an example of a **forward-reading vernier** ; in Fig. 8 is shown a corresponding **backward-reading vernier.** In this case the 10 vernier divisions have a total length equal to 11 scale divisions, and the graduation figures on the vernier run in the contrary direction to those on the scale. The reading of the scale is taken at the arrow on the vernier, and the second decimal is taken from the vernier as before.

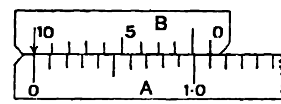

FIG. 8.—A backward-reading vernier.

The following rule is useful in the construction and reading of verniers : Let the total length of a forward-reading vernier be $(N-1)$ scale divisions, or $(N+1)$ in a backward-reading vernier, and let there be N divisions on the vernier, then the vernier reads to $1/N$ scale division.

It may be verified by the student that, if the vernier has a length of $(2N\mp 1)$ scale divisions ($-$ for forward and $+$ for backward-reading), and if there are N divisions on the vernier, then the reading may be taken to $1/N$ scale division.

**Measurement of angles.**—In Fig. 9 is shown a **protractor** by means of which angles may be measured to one minute. A semicircular piece of brass A is fitted with an arm BCD capable of rotating about a centre at D. A semicircular scale divided into half-degrees is engraved on A

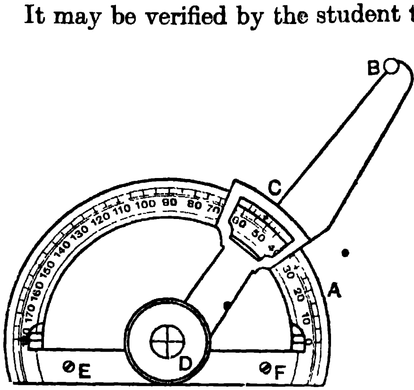

FIG. 9.—Vernier protractor.

and the arm has a vernier. The centre of the semicircular scale lies at the intersection of two cross lines ruled on a piece of glass at D. The edge BC, on being produced, passes through the zero arrow on the vernier and also through the point of intersection of the cross lines at D.

The vernier is **central-reading**, and is shown enlarged on a straight scale in Fig. 10. The total length of the vernier is 29 scale divisions,

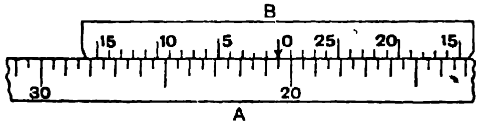


FIG. 10.—Protractor scale and vernier.

and it has 30 divisions; hence it reads to one-thirtieth of a scale division, *i.e.* to one minute. It will be seen that the zero is at the centre mark. The point on the scale to which the reading refers is usually marked by an arrow. Needles project at E and F on the under side of the instrument and prevent slipping (Fig. 9).

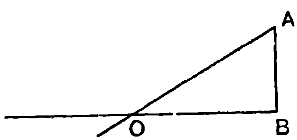


FIG. 11.

EXPT. 3. **To measure an angle.** (*a*) Draw two lines AO and BOC intersecting at O (Fig. 11). Set the protractor so that the intersection of the cross lines coincides with O and the marks 0° and 180° fall on BOC. Set the arm so that its edge BC (Fig. 9) coincides with OA. Note the reading as the magnitude of the angle AOB. A small lens will be found useful in reading the vernier.

From any point A on OA draw AB perpendicular to OB. Measure OB, BA and OA and evaluate the trigonometrical ratios:

$$\sin AOB = \frac{AB}{OA}; \quad \cos AOB = \frac{OB}{OA}; \quad \tan AOB = \frac{AB}{OB}.$$

Consult trigonometrical tables, and write down the values of the angle AOB corresponding with the calculated values of the sine, cosine and tangent. Take the mean of these values and compare it with the value found by means of the protractor.

(*b*) Draw any triangle. Measure its three angles by means of the protractor. Verify the proposition that the sum of the three angles of any triangle is equal to 180°.

**Vernier calipers.**—The vernier calipers (Fig. 12) consist of a steel bar having a scale engraved on it. Another piece may slide along the bar and carries a vernier; there is a clamp and slow-motion screw by means of which the sliding piece may be moved slowly along the bar. The article to be measured is placed between the jaws of the calipers,

and the sliding piece is brought into contact with it so as to nip it gently.

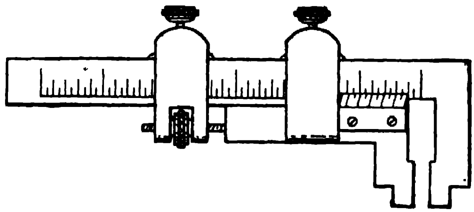

FIG. 12.—Vernier calipers.

In metric calipers the scale shows centimetres, with half-millimetre subdivisions. The vernier has a length of 24 scale divisions (*i.e.* 12 mm.) and has 25 divisions; hence the instrument reads to $\frac{1}{25} \times \frac{1}{2} = 0{\cdot}02$ mm. In British instruments the scale of inches is subdivided into fortieths of an inch. The vernier has a length of 24 scale divisions and has 25 divisions. Readings may be taken to $\frac{1}{25} \times \frac{1}{40} = 0{\cdot}001$ inch. In reading either scale, a small lens is desirable.

**Micrometer or screw-gauge.**—This instrument (Fig. 13) somewhat resembles calipers, having a screw fitted to one leg. The object to be measured is inserted between the point of the screw and the fixed abutment on the other leg, and the screw is rotated until the object is nipped gently. A scale is engraved along the barrel containing the screw, and another scale is engraved round the thimble of the screw. In Fig. 14 is shown an enlarged view of these scales. The screw has two threads per millimetre; hence one revolution will produce an axial movement of 0·5 mm. The barrel scale A shows millimetres; a supplementary scale immediately below A shows half-millimetres. The thimble scale B has 50 divisions; as one complete turn of the thimble is equivalent to 0·5 mm., one scale division movement of B past the axial line of the scale A is equivalent to $\frac{1}{50} \times 0{\cdot}5 = 0{\cdot}01$ mm. Hence readings may be taken to one hundredth of a millimetre. In Fig. 14 the scales are shown set at 7·47 mm.

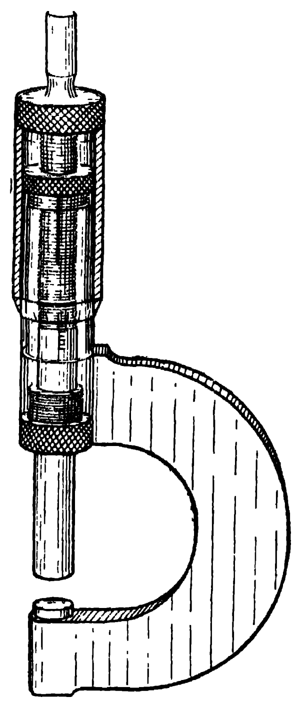

FIG. 13.—Micrometer.

In micrometers graduated on the British system the screw has usually 40 threads per

inch ; the barrel scale A shows inches divided into fortieths ; the thimble scale has 25 divisions. Hence the instrument reads to

$$\frac{1}{25} \times \frac{1}{40} = 0{\cdot}001 \text{ inch.}$$

If the point of the screw is in contact with the abutment, the scales should read zero ; if this is not so, the reading should be noted and applied as a correction to subsequent measurements.

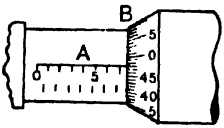


FIG. 14.—Micrometer scales.

EXPT. 4. — **Use of vernier calipers and micrometers.** Take again the bodies used in Expt. 2. Remeasure them, using the vernier calipers and the micrometer. Calculate the volumes from these dimensions, and compare the results with those obtained by the methods employed in Expt. 2.

The student is here reminded that the results of calculations should not contain a number of significant figures greater than is warranted by the accuracy of the measurements. Thus it would be absurd to state a result of 321·46934 cubic millimetres when the instrument employed reads to 0·01 mm. only. Four significant figures are sufficient for most results ; the usual plan is to state one significant figure in excess of those of which the accuracy is undoubted ; for example, 321·46 may be taken to mean that 321·4 is of guaranteed accuracy, but that there is doubt regarding the last significant figure 6.

**Spherometer.**—An ordinary type of spherometer is shown in Fig. 15. A small stool A has three pointed legs B, C and D arranged at the corners of an equilateral triangle. A micrometer screw E is fitted at the centre of the circumscribed circle of the triangle, and is pointed at its lower end. F is a graduated circular plate fixed to the screw ; there is a milled head at G for convenience in rotating the screw. A scale H is fixed to A, and has divisions cut on it at intervals equal to the pitch of the screw. The instrument rests on a glass plate K, the upper surface of which is as nearly plane as possible. L is an object the thickness of which is to be determined.

In the instrument illustrated the screw has two threads per millimetre, and the circular scale on F has 50 divisions, each subdivided into 10 ; the instrument therefore reads to

$$\frac{1}{500} \times \frac{1}{2} = 0{\cdot}001 \text{ mm.}$$

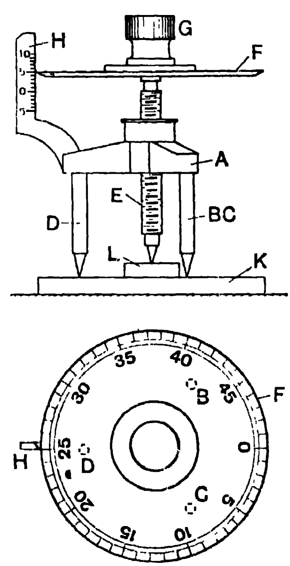


FIG. 15.—Spherometer.

EXPT. 5.—**Thickness of an object by use of the spherometer.** Place the spherometer on the plane glass plate. Rotate the screw until all four points bear equally on the glass ; this condition may be tested by pushing one of the legs in a direction nearly horizontal. If the instrument rotates, the screw point is bearing too strongly and must be raised. Should simple sliding occur, then the screw point is not bearing sufficiently. Note the readings of the scales. Unscrew E sufficiently to enable the object to be placed under the screw point, and make the adjustments as before. Read the scales again ; the difference of the two readings will give the thickness required.

Measure the thickness of the small objects supplied at three or four spots and state the average thickness of each object.

EXPT. 6.—**Use of the spherometer in determining the radius of curvature of a spherical surface.** Measure the radius of curvature of the spherical surface supplied by use of the spherometer in the following manner: Place the instrument on the plane glass plate and obtain the readings of the scales; these may be denoted the zero readings. Place the spherometer on the spherical surface (Fig. 16); adjust it and again note the scale readings. The difference between these readings will be equal to AB in Fig. 16.

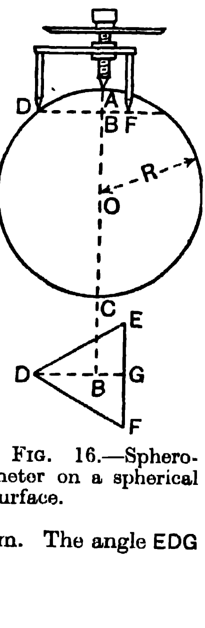


FIG. 16.—Spherometer on a spherical surface.

Let $AB = h$ millimetres,

$R =$ the radius of curvature in millimetres.

Then, from the geometry of Fig. 16, we have

$$CB \times BA = BD^2,$$

or,

$$(2R - h)h = BD^2;$$

$$\therefore\ R = \frac{BD^2}{2h} + \frac{h}{2}. \quad \text{..................(1)}$$

To obtain BD, place the spherometer on a piece of tinfoil and press gently so as to mark the positions of the three legs D, E, F (Fig. 16). Measure DE, EF and FD, and take the mean ; let this dimension be $a$ mm. The angle EDG is 30° and BD is two-thirds of DG, hence

$$BD = \frac{2}{3}DG = \frac{2}{3}DE \cos 30^\circ$$

$$= \frac{2}{3}a\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{3}a;$$

$$\therefore\ BD^2 = \frac{3}{9}a^2 = \frac{1}{3}a^2. \quad \text{..............................(2)}$$

Substitution in (1) gives :

$$R = \frac{\frac{1}{3}a^2}{2h} + \frac{h}{2}$$

$$= \frac{a^2}{6h} + \frac{h}{2}. \quad \text{..................................(3)}$$

In the case of a very flat spherical surface, $h$ will be very small ; the first term in (3) will then be very large when compared with the second term, and we may write :

$$R = \frac{a^2}{6h} \cdot \qquad (4)$$

The method of measurement and reduction is the same for both convex and concave surfaces.

Measure each of the given surfaces at two or three places ; calculate the radius of curvature for each reading, and state the mean radius.

EXPT. 7.—**Micrometer microscope.** In this instrument, the object to be measured is placed opposite B and is observed through a microscope A (Fig. 17). The microscope has a scale finely engraved on glass in the eyepiece at C, and is focussed so as to obtain sharp images of both scale and object when viewed through the eyepiece. The microscope may be traversed horizontally by means of a thumb-screw D, and may be raised or lowered in the supporting pillar by use of another thumb-screw E. The microscope carries a scale F divided in millimetres and a vernier G reading to 0·1 mm. is attached to the pillar.*

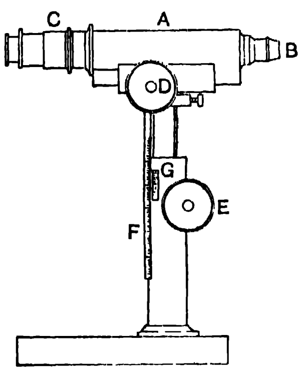

FIG. 17.—Micrometer microscope.

First obtain the value of an eyepiece scale division as follows : Focus sharply the object and scale ; move the eye slightly up and down and observe whether the object and scale as seen through the eyepiece suffer any displacement relatively to one another. If so, adjust the focussing arrangements until this movement disappears. Use E to bring zero on the eyepiece scale into coincidence with one edge or other fine mark on the object ; read and note the pillar scale and vernier. Use E to bring another mark on the eyepiece scale, say the fiftieth, into coincidence with the same mark on the object ; again read and note the pillar scale and vernier. The difference of these readings gives the value in millimetres of 50 eyepiece scale divisions ; hence calculate the value of one eyepiece scale division. Repeat the operation, using the eyepiece scale marks 20 and 70, 35 and 85, and 50 and 100. Compare the values and state the average value of an eyepiece scale division.

Measure the thickness of the objects supplied by noting the eyepiece scale marks at the top and bottom, estimating by eye to one-tenth of a scale division. Take the difference and convert into millimetres.

Measure also the bores of the glass tubes supplied.

* For the optical theory of this instrument, the student is referred to the Part of the volume devoted to Light.

**Weighing.**—The choice of a balance to be used in weighing a given body depends upon the weight of the body and also upon the accuracy required. In using balances capable of dealing with heavy bodies—up to 10 kilograms say—no special precautions need be observed other than that of placing gently both body and weights in the scale pans.

Delicate balances are fitted inside glass cases, and have arrangements by means of which the motions of the various parts may be arrested and all knife edges relieved of pressure when the balance is not in use. These arrangements are operated by a handle or lever outside the case; the handle should be moved very gently, and no weights should be placed on, or removed from either scale pan without first using the handle to arrest the motion. The sets of weights used with delicate balances are kept in partitioned boxes, and should not be fingered; forceps are provided for lifting the weights.

EXPT. 8.—**Use of a balance.** Weigh each of the bodies used in Expt. 4, thus determining its mass. Find the density of each material, making use of the equation given on p. 3, and of the volumes calculated in performing Expt. 4.

Balances are subject to errors, most of which are eliminated in the following method of weighing. Place in one of the scale pans any convenient body of weight somewhat in excess of that of the body to be weighed; add weights to the other scale pan until balance is secured; let the total weight be $W_1$. Remove the weights, and place in the empty scale pan the body to be weighed. Add weights ($W_2$ say) until balance is again restored. It is obvious that the weight of the body is equal to the difference $(W_1 - W_2)$.

EXPT. 9.—**Measurement of areas.** Draw any triangle on a piece of rectangular cardboard. Calculate the area of the triangle by use of the rules:

(i) Area = base × half the perpendicular height.

(ii) Area $= \sqrt{s(s-a)(s-b)(s-c)}$, where $s$ is half the sum of the lengths of the sides.

Copy the triangle on a piece of squared paper and find its area by counting the number of included squares. The copying of the figure may be obviated by use of a piece of squared tracing paper, covering the original figure.

Calculate the area of the whole card by taking the product of its length and breadth. Weigh the card, and calculate the weight per square centimetre by dividing the weight by the area. Carefully cut out the triangle and weigh it separately. Find the area of the triangle from:

Weight of triangle = area in sq. cm. × weight per sq. cm.

Compare the results of these methods.

**The planimeter.**—Areas may be measured by means of a planimeter (Fig. 18). This instrument consists of a bar A to which another bar B is jointed at C, so that the bars may have relative movement in a plane. B may rotate about a needle point pushed into the paper at D, and is loaded with a weight at E. A rests on a wheel F, which may roll on the paper, and has a tracing needle at G which may be carried round the boundary of the area to be measured. It may be shown that the area is proportional to the product of the distance between C and G and the distance through which the circumference of the wheel F rolls when G is carried completely round the boundary of the area.

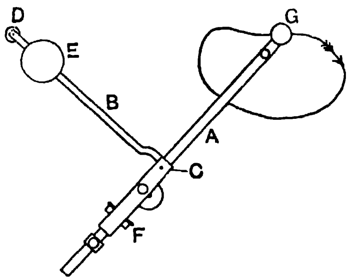

Fig. 18.—Planimeter in use.

The instrument is shown in greater detail in Fig. 19. It will be noted that the wheel has a scale engraved round its circumference;

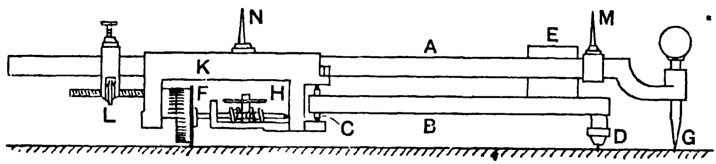

Fig. 19.—Planimeter.

there are 100 divisions on this scale, and a vernier enables the scale to be read to one-tenth of a scale division. A small indicator wheel H, driven from F, registers the number of complete revolutions of F. F, H and the joint C are carried on a bracket K, which may be clamped in any position on the bar A; a slow-motion screw L enables the distance CG to be adjusted finely. Pointers M and N are fixed to the bar A and the bracket K, and are so placed as to indicate the distance CG. Marks are placed on A to facilitate the adjusting of the positions of K suitable for measuring the area in square centimetres or square inches.

The instrument should be used on a sheet of drawing paper sufficiently large to enable the whole movements of the wheel F to be completed without coming off the paper. The surface of the paper should not be highly polished, which might lead to slipping and consequent lost motion of the wheel, nor should the surface be too rough. It is best to arrange the initial position so that the arms A and B are at right angles approximately. The tracing point G should be carried clockwise round the boundary.

Expt. 10.—**Use of the planimeter.** Draw a circle 10 cm. in diameter on the paper. Set the planimeter to the scale of square centimetres; place it on the paper with G at a mark on the circumference of the circle. Set the wheel F at zero. Carefully carry the pointer G round the boundary

and stop at the mark. Read and note the scale and vernier. Carry the pointer round a second and third time, reading the scale and vernier each time the mark is reached. Take the differences, giving three results for the area; these results should be in fair agreement.

Calculate the area of the circle from:

$$\text{Area} = \pi r^2 \text{ square cm.},$$

where $r$ is the radius of the circle in cm. Compare the calculated area with the mean area obtained by the planimeter.

EXPT. 11.—**Measurement of volumes by the displacement of water.** In Fig. 20, A is a jar containing water and fitted with a hook gauge B. The hook gauge is simply a sharp-pointed piece of wire bent to the proper shape and clamped to the side of the vessel; it is used for adjusting accurately the surface level of the water. C is the body the volume of which has to be determined. D is a graduated measuring jar having a scale of cubic centimetres engraved on its side. First adjust the water level so that the point of the hook gauge is just breaking the surface of the water. By means of a fine thread, lower carefully the body into the jar. Use a pipette to remove water until the level is restored as shown by the hook gauge. Discharge all the water removed by the pipette into the measuring jar. Read and note the volume of this water as shown by the scale; it is evident that this reading will give the volume of the body.

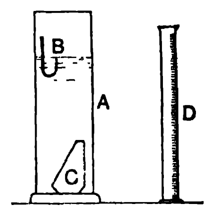

FIG. 20.—Volume by displacement.

Use this method to check the volumes of some of the larger bodies calculated in Expt. 4. The method cannot be applied with sufficient accuracy to bodies of very small dimensions, as the change in level of the water in the jar would then be inappreciable.

## Exercises on Chapter II

**1.** A scale is divided into twentieths of an inch and has to be read to one twenty-fifth of a scale division by means of a vernier. Show by sketches how to construct a suitable forward-reading vernier; also a backward-reading vernier.

**2.** The circle of an instrument used for measuring angles is divided to show degrees, and each degree is divided into six equal parts. Show how to construct a forward-reading vernier which will enable angles to be read to the nearest third of a minute. Give sketches.

**3.** A micrometer, or screw gauge, has a screw having fifty threads to an inch; the barrel scale has graduations showing fiftieths of an inch. The instrument can read to the nearest thousandth of an inch. How many divisions has the thimble scale? Show, by sketches, the scales when the instrument is reading 0·437 inch.

**4.** A spherometer has a screw with 40 threads to an inch. How many divisions should the graduated circle have if the instrument reads to 0·0001 inch?

**5.** The fixed legs of a spherometer are at the corners of an equilateral triangle of 4 cm. side. When placed on a certain spherical surface the instrument reads 5·637 mm. Find the radius of curvature of the surface. The instrument has no zero error.

**6.** The same spherometer is used on another spherical surface and reads 0·329 mm. Find the radius of curvature of the surface.

**7.** In calibrating the eyepiece scale of a micrometer microscope the following readings were taken :

| | | | | | |
|---|---|---|---|---|---|
| Eyepiece scale - | 0 | 30 | 50 | 80 | 100 |
| Pillar scale, mm. - | 35·6 | 33·6 | 32·3 | 30 3 | 29·0 |

What is the value in mm. of an eyepiece scale division?

**8.** The following dimensions of a metal frustum of a cone were measured with vernier calipers : Perpendicular height, 2·616 inches ; diameter of small end, 1·876 inches ; diameter of large end, 2 inches. The frustum was weighed and found to be 2·22 lb. Find the density of the metal in lb. per cubic inch.

**9.** Describe how you would proceed to determine by experiment the relation between the length of the circumference of any circle and its diameter. Describe any form of screw gauge you have used.

**10.** Give sketches showing the construction of any planimeter you have used. Describe how the instrument is used in determining the area of a figure having an irregular curved boundary. State any precautions which must be observed.

**11.** The micrometer screw of a spherometer, instead of having two threads per millimetre, actually has 20·01 threads per centimetre. The circular scale has 500 divisions. When placed on the plane glass plate and adjusted, the scales read 0·005 mm. An object is then measured, and the reading of the scales gives 2·642 mm. What is the actual thickness of the object?

**12.** A micrometer reads to 0·01 mm. When screwed home, the reading is 0·05 mm. The instrument was then applied to a steel ball, and the following diameters were obtained in three directions mutually perpendicular : 24·52 mm., 24·50 mm., 24·53 mm. State the mean diameter of the ball and calculate its volume.

# CHAPTER III

## DISPLACEMENT. VELOCITY. ACCELERATION

**Motion of a point.**—The motion of any body and its position at any instant may be specified by reference to chosen lines. In general, the motion of a body is complex; all points in it do not possess motions precisely alike in all respects. Hence it is convenient to commence the study of motion by the consideration of the motion of a point, or of a **particle**, *i.e.* a body so small that any differences in the motions of its parts may be disregarded.

In **rectilinear motion**, or motion in a straight line, it is sufficient to consider as fixed in space the line in which the point is moving. The position at any instant of a point P moving along the straight line OA (Fig. 21) may be specified by stating the distance OP from a fixed point O in the line; O may be called the origin.

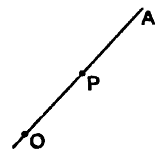

FIG. 21.—Rectilinear motion.

In **uniplanar motion** the point has freedom to move in a given plane. The position and motion of the point at any instant may be referred conveniently to any two fixed lines, mutually perpendicular, and lying in the plane of the motion; such lines are called **coordinate axes**. Thus in Fig. 22 a point P is describing a curve in the plane of the paper, supposed to be fixed in space. Its precise position at any instant may be defined by stating the perpendicular distances $y$ and $x$ from the two coordinate axes OX and OY. It will be noted that OX and OY divide the space surrounding the origin O into four compartments. Useful conventions are to describe $x$ as positive or negative according as P is situated on the right or left of OY. Similarly, $y$ is positive or negative according as P is above or below OX.

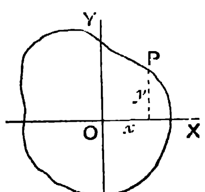

FIG. 22.—Motion in a plane.

More complex states of motion arise when the moving point is not confined to one plane; for example, a person ascending a spiral staircase. Most of these cases are beyond the scope of this book.

**Illustration of rectilinear and uniplanar motion.**—The mechanism shown in Fig. 23 consists of a **crank** CB capable of revolving about an axis at C perpendicular to the plane of the paper. A **connecting rod** AB is jointed to the crank at B by means of a pin and also to a block D

capable of sliding in a slot in the frame E. If the crank is revolving, the block D has rectilinear motion to and fro in the slot, and B has circular motion in the plane of the paper.

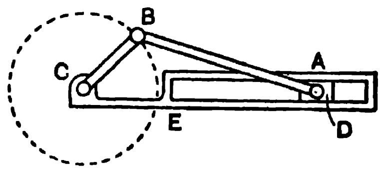


FIG. 23.—Slider-crank mechanism.

**Locus of a moving point.**—The determination of the position at any instant of a point in the above-mentioned and similar mechanisms may be made the subject of mathematical calculation. A more useful method employed in practice is to draw the **locus**, or path of the moving point; such a path will show the positions of the point throughout the whole range of possible movement of the mechanism.

An illustration of the method is given in Fig. 24, which shows the locus of a point D on the connecting rod of a mechanism similar to that given in

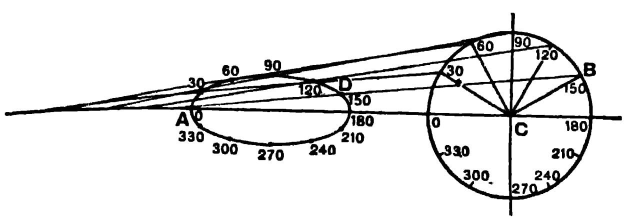

FIG. 24.—Locus of a point in a connecting rod.

Fig. 23. Outline drawings of the crank CB and connecting rod BA are constructed for successive positions of the crank, differing by 30° throughout the entire revolution. (For the sake of clearness, the positions above CA alone are shown in Fig. 24.) The position of D along AB is marked carefully on each drawing; a fair curve drawn through these points will give the required locus.

**Displacement.**—Suppose that a point occupies a position A at a certain instant (Fig. 25), and that at some other instant its position is B. Draw the straight line AB; AB is called the **displacement** of the point. In making this definition the precise path by which the point travelled from A to B is immaterial. For example, the point might have first a displacement from A to C, and then from C to B, with exactly the same change in position as would occur by travelling directly along the straight line AB. Hence we may say that the displacement AB is

equivalent to the displacements AC and CB. AB is called the **resultant displacement**, and AC and CB are **component displacements.**

It is evident that the number of component displacements may be unlimited. Thus, in Fig. 26, the components AC, CD, DE, EF, FG

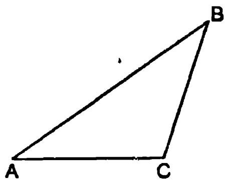


FIG. 25.—Triangle of displacements.

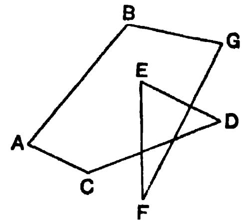


FIG. 26.—Polygon of displacements.

and GB successively applied to the point, are equivalent to the resultant displacement AB.

**Specification of a displacement.**—In stating a displacement it is necessary to specify (*a*) the initial position, (*b*) the direction of the line in which the point moves, (*c*) the **sense** of the motion, *i.e.* from A towards B or *vice versa* (Fig. 25), (*d*) the magnitude of the displacement.

The sense may be indicated by the order of the letters defining the initial and final positions in a displacement, AB or BA, or by placing an arrow point on the line.

**Vector and scalar quantities.**—Any physical quantity which requires a direction to be stated in order to give a complete specification is called a **vector quantity**; other quantities are called **scalar quantities.** Displacement and force are examples of vector quantities; mass, density and volume are scalar quantities. Any vector quantity may be represented by drawing a straight line in the proper direction and sense.

The operations performed in Figs. 25 and 26 are examples of the **addition of vectors.** The operation consists in constructing a figure in which a straight line is drawn from the initial position to represent the first vector, making the line of a length to represent to scale the magnitude of the quantity, and drawing it in the proper direction and sense. From the end of this line remote from the initial position, another line is drawn in a similar manner to represent the second vector, and so on until all the components have been dealt with. The resultant vector will be represented by the line which must be drawn from the initial position in order that the completed figure may be a closed polygon.

Fig. 25, in which there are two component vectors only, may be called the **triangle of displacements**; the name **polygon of displacements** may be given to Fig. 26.

**Velocity.**—The **velocity** of a moving point may be defined as **the rate of change of position in a given direction**; the time taken, the distance travelled, and the direction of motion are all taken into account in stating a velocity. Velocity is a vector quantity. In cases where the direction of motion does not require to be considered, the term **speed** is employed to express the rate of travelling.

Velocity may be **uniform**, in which case the point describes equal distances in equal intervals of time; the velocity is said to be **variable** if this condition be not complied with.

The velocity at any instant of a point having uniform velocity may be measured by stating the distance travelled in unit time. Thus, if a total distance s be described in t seconds, then the magnitude of the velocity v at any instant is given by

$$v = \frac{s}{t} \cdot \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(1)$$

This will be in cm. per sec., or feet per second, according as $s$ is in cm. or feet. The specification of the velocity given by (1) is completed by stating also the direction of the line in which motion takes place and the sense of the motion along this line.

In the case of a variable velocity, the result given by use of equation (1) is the average value of the velocity during the interval of time $t$. Thus the average velocity of a train which travels a total distance of 400 miles in 8 hours (including stops) is 400÷8, or 50 miles per hour.

The dimensions of velocity are $l/t$ or $lt^{-1}$.

If a point moves with variable velocity, the velocity at any instant may be stated as the distance which the point would travel during the succeeding second if the velocity possessed at the instant under consideration remained constant. It is also the limit reached by the quantity $\frac{ds}{dt}$ when $ds$ and $dt$, the changes in distance and time, are vanishingly small.

$$v = \frac{ds}{dt}.$$

**Acceleration.**—**Acceleration** means **rate of change of velocity**, and involves both change of velocity and the time interval in which the change has been effected. Acceleration is measured by stating the change of velocity which takes place in unit time. Unit acceleration is possessed by a particle when unit change in velocity occurs in unit time. Applying the method of small changes, as above, $a = \frac{dv}{dt}$.

EXAMPLE. At a certain instant, a particle having rectilinear motion has a velocity of 25 cm. per sec. The velocity is found to increase uniformly during the succeeding 5 seconds to 60 cm. per sec. Find the acceleration.

Increase in vel. in 5 secs. $= 60 - 25 = 35$ cm. per sec.

" " 1 sec. $= \frac{35}{5} = 7$ cm. per sec.

Hence the acceleration is 7 cm. per second in every second, or, as is usually stated, 7 cm. per sec. per sec., or 7 cm./sec.$^2$

It will be noted that time enters *twice* into the statement of a given acceleration, once in expressing the change in velocity, and again in expressing the time interval in which the change was effected.

Acceleration may be **uniform**, in which case equal changes in velocity occur in equal intervals of time. Otherwise the acceleration is **variable**. In the case of uniform acceleration, the acceleration at any instant is calculated by dividing the total change in velocity by the time in which the change takes place. A similar calculation made for the case of variable acceleration gives the average acceleration during the time interval considered.

Since acceleration involves velocity, it is a vector quantity. To specify completely a given acceleration, the magnitude, the line of direction and the sense of the acceleration along the line of direction must be stated.

The dimensions of acceleration are obtained by dividing the dimensions of velocity by time, giving $l/t \div t = l/t^2$ or $lt^{-2}$.

**Displacement, velocity and acceleration graphs.**—A convenient method of studying questions involving displacement, velocity and acceleration is to construct graphs in which the magnitudes of these quantities are plotted as ordinates and the time intervals as abscissae.

EXAMPLE 1.—A point P, travelling in a straight line OA, passes through the origin O at a certain instant, and has a uniform velocity of 40 cm. per sec. Plot displacement-time and velocity-time graphs.

Since the velocity $v$ is uniform, the displacement in any interval of time $t$ seconds is given by $s = vt$.

| | | | | | |
|---|---|---|---|---|---|
| Time $t$ secs., reckoned from O, - - | 0 | 1 | 2 | 3 | 4 |
| Displacement $s$ cm., reckoned from O, | 0 | 40 | 80 | 120 | 160 |

These numbers, plotted as shown in Fig. 27, give a straight line displacement-time graph OB.

The velocity is uniform, therefore the velocity-time graph CD is parallel to the time axis (Fig. 28). In this graph, since OC represents the constant velocity $v$, and O4 represents time $t$, the rectangular area C4 represents the product $vt$, and hence represents the displacement in the time $t$.

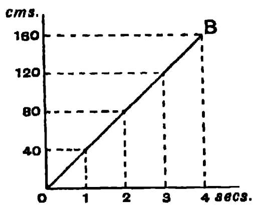


FIG. 27.

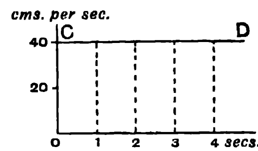


FIG. 28.

Graphs for uniform velocity.

EXAMPLE 2.—A point P, travelling in a straight line OA, is at rest in the initial position O and has a constant acceleration of 20 cm. per sec. per sec. Plot velocity-time, acceleration-time and displacement-time graphs.

It is evident that the point will have a velocity of 20 cm. per sec. at the end of the first second, and that its velocity increases by 20 cm. per sec. during each second of the motion ; hence the velocity $v$ at the end of any time interval $t$ seconds may be found from $v=20t$ cm. per sec.

| | | | | | |
|---|---|---|---|---|---|
| Time $t$ secs., reckoned from O, - - | 0 | 1 | 2 | 3 | 4 |
| Vel. $v$ cm. per sec., reckoned from O, - | 0 | 20 | 40 | 60 | 80 |

Plotting these numbers as shown in Fig. 29, we obtain a straight-line velocity-time graph OE.

The acceleration-time graph is shown in Fig. 30. Since the acceleration is uniform, it follows that the graph FG is parallel to the time axis.

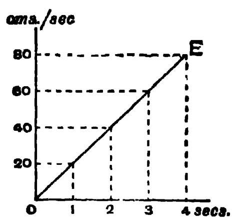


FIG. 29.

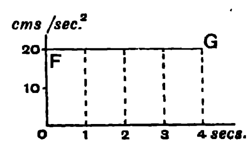


FIG. 30.

Graphs for constant acceleration, starting from rest.

The displacement during any interval of time is given by the product of the average velocity during the interval and the interval of time. Thus, if $v_a$ be the average velocity in cm. per sec. during a time $t$ seconds, then the displacement $s$ in cm. is equal to the product $v_a t$.

For example, when the point is at the origin the velocity is zero, and at the end of the first three seconds the velocity is 60 cm. per second (Fig. 29); hence the average velocity during the first three seconds is given by

$$v_a = \frac{0+60}{2} = 30 \text{ cm. per sec.}$$

And $s = 30 \times 3 = 90$ cm. in the first three seconds.

| | | | | | |
|---|---|---|---|---|---|
| Time interval in secs., reckoned from O, - | 0 | 1 | 2 | 3 | 4 |
| $v_a$ during time interval, cm. per sec., - | 0 | 10 | 20 | 30 | 40 |
| Displacement during interval, cm., - | 0 | 10 | 40 | 90 | 160 |

Plotting displacements and time as shown in Fig. 31, we obtain the curved graph OH, which shows the relation of displacement and time.

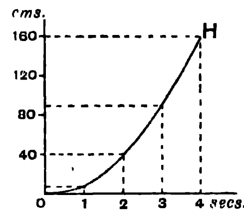

FIG. 31.—Displacement graph.

The student will note that the average velocity during any time interval in Fig. 29 is represented by the average height of the portion of the graph inclosed by the ordinates at the beginning and end of the interval. The distance between the feet of the ordinates represents time, hence the area of the graph represents the product of average velocity and time, and therefore represents the displacement during the interval.

**Equations for rectilinear motion.**—The following equations are for simple cases which occur frequently, and are deduced from the velocity-time graphs.

**Case 1.** **Uniform velocity.**—This case has been dealt with on p. 22, and the following equation was deduced:

$$s = vt. \quad \text{.......................................(1)}$$

**Case 2.** **Uniform acceleration, starting from rest.**—Let the acceleration be $a$. The velocity at the end of the first second will be equal to $a$, and a velocity equal to $a$ will be added during each subsequent second (Fig. 32). Hence, at the end of $t$ seconds, we have

$$v = at. \quad \text{.......................................(2)}$$

The average velocity during the first $t$ seconds is

$$v_a = \tfrac{1}{2}v = \tfrac{1}{2}at. \quad \text{.................................(3)}$$

Therefore the displacement during the first $t$ seconds is

$$s = v_a t = \tfrac{1}{2}vt, \quad \text{.................................(4)}$$

$$= \tfrac{1}{2}at \times t = \tfrac{1}{2}at^2. \quad \text{............................(5)}$$

From (2), $$t=\frac{v}{a}, \text{ or } t^2=\frac{v^2}{a^2}.$$

Substitution in (5) gives $$s=\tfrac{1}{2}a\frac{v^2}{a^2}=\frac{v^2}{2a},$$

or $$\mathbf{v^2=2as.} \quad \text{.......................................(6)}$$

**Case 3. Uniform acceleration, and starting from the initial position with a velocity u.**—The velocity-time graph is given in Fig. 33. Since the

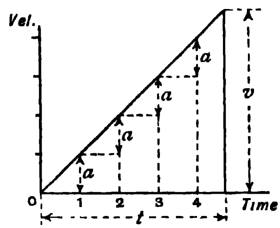


FIG. 32.—Uniform acceleration, starting from rest.

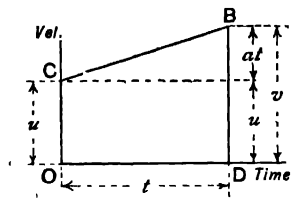


FIG. 33.—Uniform acceleration, initial velocity $u$.

initial velocity is $u$, and a velocity equal to $a$ is added during each second, the velocity at the end of $t$ seconds is

$$v=u+at,$$

or $$\mathbf{v-u=at.} \quad \text{.......................................(7)}$$

In Fig. 33, BD represents $v$ and CO represents $u$; the average velocity during the first $t$ seconds is

$$v_a=\frac{u+v}{2}=\frac{u+u+at}{2};$$

$$\therefore\ \mathbf{v_a=u+\tfrac{1}{2}at.} \quad \text{.......................................(8)}$$

The displacement during the first $t$ seconds is given by

$$s=v_at=(u+\tfrac{1}{2}at)t;$$

$$\therefore\ \mathbf{s=ut+\tfrac{1}{2}at^2.} \quad \text{.......................................(9)}$$

It will be noted that the first term in (9) gives the displacement which would have occurred had the velocity $u$ been preserved uniform throughout; the second term gives the displacement which would have taken place had the point started from rest with a uniform acceleration $a$.

An alternative method of obtaining these equations is as follows:

Since $v=\dfrac{ds}{dt}$, (p. 22), $ds=v\,.\,dt$.

Integrating in the usual way, $s=vt+C_1$, where $C_1$ is a constant determined by the starting-point of the motion. If the body starts

from the origin of measurements, $s=0$ when $t=0$, and it follows that $C_1=0$,

$$\therefore\ s=vt. \quad\text{.................................}(1)$$

Again, $a=\dfrac{dv}{dt}$, (p. 22), $\qquad \therefore\ dv=a\,.\,dt,$

and on integrating, $\qquad v=at+C_2.$

If the body starts with velocity $u$, $v=u$ when $t=0$,

$$\therefore\ C_2=u \quad\text{and}\quad v=u+at. \quad\text{.......................}(7)$$

Since $ds=v\,dt$ and $v=u+at$,

$$ds=(u+at)dt$$
$$=u\,dt+at\,dt.$$

Integrating, $\qquad s=ut+\tfrac{1}{2}at^2+C_3,$

and, as before, if $s=0$ when $t=v$, then $C_3=0$,

$$s=ut+\tfrac{1}{2}at^2. \quad\text{.................................}(9)$$

From (7), $\qquad t=\dfrac{v-u}{a}$, or $t^2=\dfrac{(v-u)^2}{a^2}$.

Substitute in (9), giving

$$s=u\left(\frac{v-u}{a}\right)+\tfrac{1}{2}a\,\frac{(v-u)^2}{a^2}$$
$$=\frac{uv-u^2}{a}+\frac{v^2+u^2-2uv}{2a}$$
$$=\frac{2uv-2u^2+v^2+u^2-2uv}{2a}$$
$$=\frac{v^2-u^2}{2a},$$

or $$v^2-u^2=2as. \quad\text{.................................}(10)$$

**Bodies falling freely.**—Experiment shows that any body falling freely under the action of gravitation has uniform acceleration. The term **freely** is used to indicate that the resistance of the atmosphere has been removed, or has been neglected. The symbol $g$ is used to denote the acceleration of a body falling freely. All the equations obtained in Cases 2 and 3 above may be employed by substitution of $g$ for $a$, and the height $h$ for $s$. Thus equations (6) and (10) will read respectively :

$$v^2=2gh. \quad\text{..............................}(a)$$
$$v^2-u^2=2gh. \quad\text{..............................}(b)$$

($a$) applies to a body falling freely from rest, and may be used in calculating the velocity at the end of a fall from a height $h$. ($b$) applies to a similar case in which the body is projected downwards with an initial velocity $u$ ; the terminal velocity $v$ may be calculated from ($b$).

**Variations in the value of g.**—The value of $g$ varies somewhat at different parts of the earth; in Britain, 981 cm. per sec. per sec., or 32·2 feet per sec. per sec. may be used in most calculations. The value of $g$ at any given place depends upon the distance between that place and the centre of the earth. The value of $g$ at sea-level in latitude 45° is sometimes chosen as a standard of reference; the value at other places depends upon the height above sea-level and also upon the latitude. Latitude is a factor on account of (1) the shape of the earth, which, being flattened somewhat towards the poles, causes sea-level at the poles to be nearer to the centre of the earth than sea-level at the equator; (2) the variation of centrifugal action with distance from the equator.

Let $g_0$ = the value of $g$ at sea-level in latitude 45°, cm. per sec. per sec.

$g$ = the value of $g$ at an elevation H metres in latitude $\lambda$, cm. per sec. per sec.

Then $g = g_0(1 - 0{\cdot}0026 \cos 2\lambda - 0{\cdot}0000002\text{H})$.

**Conventions regarding signs.**—In considering a point P moving in a straight line AB, it is convenient to choose one sense, say from A towards B, and to call velocities and accelerations having this sense **positive**; velocities and accelerations having the contrary sense will then be called **negative**. This convention enables graphs to be drawn in representation of such cases as that of a body projected upwards, coming gradually to rest, and then descending.

In Fig. 34 is given a velocity-time diagram illustrating this case. The body was projected upwards with a (positive) velocity $u$, represented by OA drawn above the time axis. Velocity is abstracted at a uniform rate $g$, and the body comes to rest, as indicated at B, at the end of a time interval represented by OB. Thereafter its velocity is downwards (negative) and increases numerically until it reaches the level of the initial position, when it possesses a negative velocity $v_2$ represented by DC. As the graph ABC is a straight line owing to the acceleration being uniform, the time taken in ascending is equal to the time of descent, and the terminal velocity $v_2$ is numerically equal to the initial velocity $u$. It is, of course, assumed that the resistance of the atmosphere is neglected throughout.

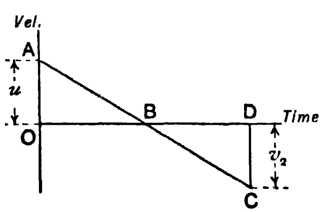

FIG. 34.—Velocities having opposite senses.

Calling downward accelerations negative, it will be noted that the acceleration in the above illustration is negative throughout the motion. During the ascent the velocity and acceleration have contrary signs, and the velocity is diminishing; during the descent the signs are similar and the velocity is increasing.

**More general case of a velocity-time graph.**—In Fig. 35 is shown a velocity-time-graph OAB. Consider two points $P_1$ and $P_2$ lying fairly close together on the graph. $P_1M_1$ and $P_2M_2$ represent velocities $v_1$ and $v_2$ respectively, and these velocities are possessed by the moving point at the end of time intervals $t_1$ and $t_2$, represented by $OM_1$ and $OM_2$ respectively. Draw $P_1K$ parallel to the time axis. Then

$$P_2K = P_2M_2 - P_1M_1 = v_2 - v_1,$$

and is the change in velocity during a time interval

$$M_1M_2 = OM_2 - OM_1 = t_2 - t_1.$$

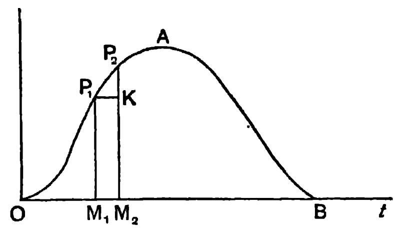


FIG. 35.—General case of a velocity-time graph.

Hence the average acceleration during this interval is given by

$$a_a = \frac{v_2 - v_1}{t_2 - t_1} = \frac{P_2M_2 - P_1M_1}{OM_2 - OM_1}.$$

The quantities required for making the calculation of the average acceleration may be scaled from the graph. There is no great error in assuming that the average acceleration so calculated is the actual acceleration at the middle of the interval $M_1M_2$. Hence an acceleration-time graph may be constructed by calculating the average accelerations as above explained for intervals throughout OB, and erecting ordinates at the middle of each interval to represent these accelerations.

It will be noted that if $P_2M_2$ is less than $P_1M_1$, the velocity is diminishing, and the acceleration has then the sign contrary to that which has been assumed for increasing velocity.

Again, the average velocity during the interval $M_1M_2$ is

$$\tfrac{1}{2}(P_1M_1 + P_2M_2) = \tfrac{1}{2}(v_1 + v_2).$$

Since the time interval is $M_1M_2 = t_2 - t_1$, it follows that the displacement during the interval $M_1M_2$ is

$$\tfrac{1}{2}(v_1 + v_2)(t_2 - t_1) = \tfrac{1}{2}(P_1M_1 + P_2M_2)M_1M_2 = \text{the area } P_1M_1M_2P_2$$

very nearly. The closeness of the approximation becomes more perfect if the interval $M_1M_2$ be made smaller and smaller, and is absolutely perfect if the interval be made indefinitely small. It is evident that the area of any similar strip of the graph will represent the displacement in the time interval represented by the base of the strip. Hence, for the total time OB, the total displacement is represented by the total area of the graph, and may be found by applying the rules of mensuration. Thus, find the area of the graph, using a planimeter, say. Divide this area by the length OB so as to find the average height of the graph. Multiply the average height by the scale of velocity, thus

obtaining the average velocity for the whole graph. Multiply the average velocity by OB, expressed in seconds. The result gives the total displacement.

## Exercises on Chapter III

**1.** A flat board has two grooves cut in it, running right across the board and intersecting at right angles at the centre of the board. A rod AB 2·5 inches long moves in the plane of the board, A being constrained to move in one groove and B in the other. Draw the locus ($a$) of a point at the centre of AB, ($b$) of a point in the rod 0·75 inch from B.

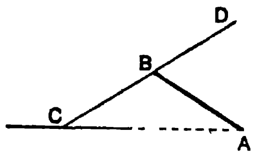


FIG. 36.

**2.** In Fig. 36, AB is a rod 2 inches long and can revolve about a fixed centre at A. CBD is another rod, jointed to AB at B, and having the end C constrained to remain in a groove, the direction of which passes through A ; CB is 2 inches long. Draw the locus of D ($a$) if BD is 2 inches long, ($b$) if BD is 3 inches long.

**3.** A point is given two component displacements, one of 24 cm. towards the north-east, and another of 30 cm. towards the north. Find the resultant displacement.

**4.** Draw a horizontal line OX as a reference axis. Starting from O, a point has the following component displacements, impressed successively : 2 inches at 30° to OX ; 3 inches at 45° to OX, 5 inches at 240° to OX ; 4 inches at 90° to OX. Find the resultant displacement.

**5.** What is the average speed in feet per second of a horse which travels a distance of 11 miles in 1·25 hours?

**6.** An observer notes that the peal of thunder is heard 3·5 seconds after seeing the flash of lightning. If sound travels at a speed of 1100 feet per second, find the distance in miles between the flash and the observer.

**7.** Two runners, A and B, start from the same place. A starts 30 seconds before B and runs at a constant speed of 8 miles per hour. B travels along the same road with a constant speed of 10 miles per hour. At what distance from the starting-point will B overtake A?

**8.** Two trains, A and B, travelling in opposite directions, pass through two stations 1·5 miles apart at the same instant. If A has a constant speed of 40 miles per hour, find the constant speed of B so that the trains shall pass each other at a distance of 0·9 mile from the station which A passed through.

**9.** A train travelling at uniform speed passes two points 480 feet apart in 10 seconds. Find the speed in miles per hour.

**10.** A train starts from rest and gains a speed of 10 miles per hour in 15 seconds. Find the acceleration in foot and second units. Sketch a velocity-time graph.

**11.** A ship travelling at 22 kilometres per hour has its speed changed to 18 kilometres per hour in 40 seconds. Find the acceleration in metre and second units. Sketch a velocity-time graph.

**12.** A body travelling at 800 feet per minute is brought to rest in $\frac{1}{8}$ second. Assume the acceleration to be uniform, and find it. Sketch a velocity-time graph.

**13.** Express an acceleration of 60 miles per hour per minute, in metres per second per second.

**14.** A train starts from rest with an acceleration of 1·1 feet per sec. per sec. Find its speed in miles per hour at the end of 25 seconds. What distance does it travel in this time? Sketch a velocity time-graph.

**15.** A train changes speed from 60 to 50 miles per hour in 15 seconds. Find the distance travelled in this interval. Sketch a velocity-time graph.

**16.** A train starts from rest with an acceleration of 0·9 foot per sec. per sec. and maintains this for 30 seconds. Constant speed is then maintained until a certain instant when steam is shut off and the brakes are applied, producing a negative acceleration of 1·5 feet per sec. per sec. until the train comes to rest. If the total distance travelled is 2 miles, find the time during which the speed was uniform and the total time for the whole journey. Sketch a velocity-time graph.

**17.** What acceleration must be given to a train travelling at 30 miles per hour in order to bring it to rest in a distance of 200 yards? Sketch a velocity-time graph.

**18.** A body falls freely from a height of 50 metres. Find the velocity just before touching the ground and the time taken. Sketch velocity-time and distance-time graphs. Take $g=981$ cm. per sec. per sec.

**19.** A stone is projected vertically upwards. Find the initial velocity in order that it may reach a height of 150 feet. If the stone falls to the original level, find the total time of flight. Sketch a velocity-time graph. Take $g=32{\cdot}2$ feet per sec. per sec.

**20.** A stone is dropped down a well and is observed to strike the water in 2·5 seconds. Find the depth of the well to the surface of the water. Take $g=32{\cdot}2$ feet per sec. per sec.

**21.** Suppose in Question 20 that the sound of the splash is heard 2·6 seconds after dropping the stone; find the depth to the surface of the water. Assume that sound travels at 1100 feet per second, and that $g=32{\cdot}2$ feet per sec. per sec.

**22.** A stone is thrown vertically downwards with a velocity of 20 feet per sec. Find the velocity at the end of the third second. What distance does it travel up to this instant?

**23.** A stone is projected vertically upwards with a velocity of 160 feet per second. Two seconds later a second stone is projected vertically from the same point. Find to what height the first will rise, and the velocity with which the second must be projected for it to strike the first as the first is just about to descend. L.U.

**24.** A stone is dropped from a height of 64 feet, while at the same instant a second stone is projected from the earth immediately below with sufficient velocity to enable it to ascend 64 feet. Find when and where the stones will meet. L.U.

**25.** Eight bodies are dropped in succession from a height at intervals of half a second. Taking $g=32$ ft. per sec. per sec., calculate and show on a diagram the positions of the bodies at the instant at which the last is dropped. What is the relative velocity of any one of the bodies to the next succeeding one? L.U.

**26.** A body moves along a straight line with varying velocity, and a curve is constructed in which the ordinate represents the velocity at a time represented by the abscissa. Prove that the distance travelled by the body in any interval is measured by the area between the two corresponding ordinates. The body is observed to cover distances of 12, 30 and 63 yards in three successive intervals of 4, 5 and 7 seconds. Can it be moving with uniform acceleration? L.U.

**27.** From the top of a tower, 75 feet high, a stone is projected vertically upwards with a velocity of 64 feet per second. Calculate its greatest elevation, its velocity at the moment it strikes the ground, and the time it takes to reach the ground. ($g=32$.)

**28.** Establish the formula $s=ut+\frac{1}{2}at^2$.

From an elevated point A a stone is projected vertically upwards. When the stone reaches a distance $h$ below A its velocity is double what it was at a height $h$ above A. Show that the greatest height attained by the stone above A is $\frac{5}{3}h$.

**29.** Two trains, A and B, leave the same station on parallel lines of way. The train A starts with uniform acceleration of $\frac{1}{2}$ foot per second per second, and attains a maximum speed of 15 miles per hour, when steam is reduced so as to keep the speed constant. B leaves 40 seconds after A with uniform acceleration of 1 foot per second per second, and attains a maximum speed of 30 miles per hour. At what distance from the station will B overtake A?

**30.** Plot a velocity-time graph from the following particulars: Draw a horizontal line OX, 5 inches in length, and divide it into 10 equal parts; each part represents 0·2 second. Draw OY perpendicular to OX, and on it construct a scale of velocities in which 0·5 inch represents 10 feet per second. The velocities in feet per second at the beginning of the time intervals shown on OX are as follows: 0, 16, 30, 42, 49, 49, 47, 40, 28, 14, 0.

(*a*) Find the change in velocity and the average acceleration during each interval of time; draw an acceleration-time graph by plotting the average accelerations at the centres of the time intervals. Scale for accelerations, one inch represents 20 feet per second per second.

(*b*) Find the average velocity during each time interval, and calculate the displacement during each interval; hence calculate the total displacement during the 2 seconds represented by OX.

**31.** If the constant of gravitation in the C.G.S. system is $6{\cdot}7\times10^{-8}$, the mean radius of the earth is $6{\cdot}4\times10^{8}$ cm., and the mean density of the earth is 5·5, calculate the acceleration due to gravity at the earth's surface, assuming that the earth behaves as if its mass were concentrated at its centre. C.W.B., H.C.

**32.** A point moves on a line in such a way that its velocity $v$ at time $t$ is given by the following table:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $t$ (secs.) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $v$ (in./sec.) | 9 | 29 | 55 | 62 | 68 | 63 | 55 | 52 | 40 | 32 | 30 |

Plot these values on a suitable graph and deduce the space-time graph of the motion. C.W.B., H.C.

**33.** A graph is obtained by plotting the reciprocal of the velocity ($v$) of a moving point against the distance ($s$) measured from some fixed point. Prove that the time taken to travel between two given distances is repre-

sented by the area between the graph, the axis of $s$ and the two ordinates of the $1/v$ curve corresponding to the two given distances.

If $s$ and $v$ are given by the following table,

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $s$ ft. - - | ·25 | 1 | 1·44 | 4 | 6·25 | 8·41 | 10·24 |
| $v$ ft./sec. - | 1 | 2 | 2·4 | 4 | 5 | 5·8 | 6·4 |

find the time between the distances 1 ft. and 8·41 ft. J.M.B., H.S.C.

**34.** The distance from a certain point of a train along a track at intervals of one minute has the following values in thousands of feet:

0, 2·43, 4·79, 7·67, 9·85, 11·88, 13·74.

Find as accurately as you can: (*a*) the speeds in feet per second at the end of one minute and at the end of 5 minutes; (*b*) the time taken to travel two miles. J.M.B., H.S.C.

# CHAPTER IV

## COMPOSITION AND RESOLUTION OF VELOCITIES AND ACCELERATIONS

**Composition and resolution of velocities.**—Velocity being a vector quantity may be represented by a straight line in the same manner as displacement. A given velocity may be regarded as made up of two or more component velocities, which may be compounded to obtain the resultant velocity by the methods of vector addition employed in Figs. 25 and 26 (p. 21). Thus, if a point has a velocity represented in magnitude, direction and sense by AB (Fig. 37), and if its initial position be A, then it will travel from A to B in one second. Suppose on arrival at B that the initial velocity of the point is suppressed, and that another velocity is imparted to it, represented by BC. The point will now travel from B to C in one second. Had both velocities been imparted simultaneously to the point when at A, the point would travel along the line AC and would arrive at C in one second. Hence AC represents the resultant velocity of which AB and BC represent the components. Similar reasoning may be applied to a number of component velocities. The triangle ABC in Fig. 37 may be called the **triangle of velocities.**

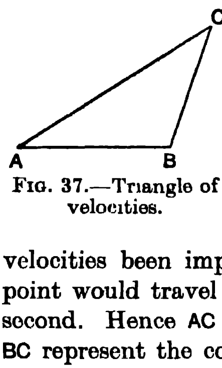

FIG. 37.—Triangle of velocities.

**Composition of velocities** is the process of finding the resultant velocity from given components; **resolution of velocities** is the inverse process.

**Parallelogram of velocities.**—Instead of a triangle of velocities ABC (Fig. 37), a construction called the **parallelogram of velocities** may be employed. In Fig. 38, a point A has component velocities $v_1$ and $v_2$ represented by AB and AC respectively. Complete the parallelogram ABDC, when the diagonal AD will represent completely the resultant velocity $v$. It is evident that the triangle ABD, which is one-half of the parallelogram, is a triangle of velocities corresponding with the triangle ABC in Fig. 37. The component velocities must be arranged, prior to constructing the parallelogram of velocities, so that the senses are either both away from A or both towards A. This process is illus-

trated in Fig. 39, in which AB and CA represent the given velocities $v_1$ and $v_2$. AC′ is drawn to represent $v_2$, and $v_1$ and $v_2$ have now senses

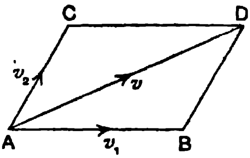


FIG. 38.—Parallelogram of velocities.

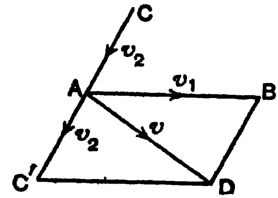


FIG. 39.

both away from A. The parallelogram is constructed as before, giving the resultant velocity $v$ represented by AD.

**Rectangular components of a velocity.**—It is often convenient in the solution of problems to use components of a given velocity taken along two rectangular axes which intersect at a point on the line of the given velocity and lie in the same plane. In Fig. 40, OA represents the given velocity $v$; OX and OY are rectangular axes complying with the above conditions. The component velocities $v_x$ and $v_y$ are found by drawing AB and AC perpendicular to OX and OY respectively, when OB and OC represent $v_x$ and $v_y$ respectively. Let the angle XOA be $\alpha$, then:

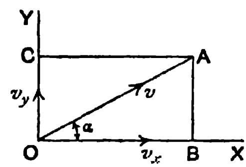


FIG. 40.—Rectangular components of $v$.

$$\frac{\mathrm{OB}}{\mathrm{OA}} = \cos\alpha, \quad \text{or} \quad \frac{v_x}{v} = \cos\alpha;$$

$$\therefore \; v_x = v\cos\alpha. \qquad (1)$$

Also,
$$\frac{\mathrm{OC}}{\mathrm{OA}} = \sin \mathrm{OAC} = \sin\alpha, \quad \text{or} \quad \frac{v_y}{v} = \sin\alpha;$$

$$\therefore \; v_y = v\sin\alpha. \qquad (2)$$

Further,
$$\mathrm{OA}^2 = \mathrm{OB}^2 + \mathrm{BA}^2 = \mathrm{OB}^2 + \mathrm{OC}^2,$$

or
$$v^2 = v_x^{\,2} + v_y^{\,2};$$

$$\therefore \; v = \sqrt{v_x^{\,2} + v_y^{\,2}}. \qquad (3)$$

**Relative velocity.**—A person standing on the earth and watching a moving body does not perceive the absolute motion of the body; what he does observe may be described as the motion of the body **relative to the earth.** In such cases it is convenient to regard the earth, and hence the observer, as fixed in space. The velocity of one body relative to another may be defined as **that velocity which an observer, situated on and moving with the second body, would perceive in the first.**

EXAMPLE.—Suppose two trains to be moving on parallel lines of railway and to have equal velocities of like sense. A passenger in either train would perceive no velocity whatever in the other train, which would appear to him to be at rest. The velocity of either train relative to the other train is zero in this case. If one train A has a velocity of 40 miles per hour towards the north, and the other B a velocity of 30 miles per hour also towards the north, an observer in B will see A passing him at 10 miles per hour, and would describe the velocity of A relative to B as 10 miles per hour towards the north. An observer in A would see B falling behind at 10 miles per hour and would describe the velocity of B relative to A as 10 miles per hour towards the south.

These statements may be generalised by saying that **the velocity of one body A, relative to another body B, is equal and opposite to the velocity of B, relative to A.**

**Determination of relative velocity.**—In Fig. 41 a point A has a velocity $v_A$ relative to the paper and represented by AC. Another point B has

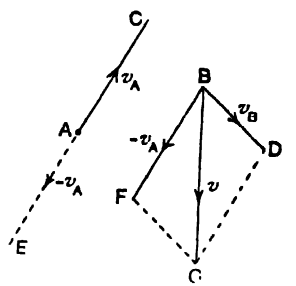


FIG. 41.—Velocity of B relative to A.

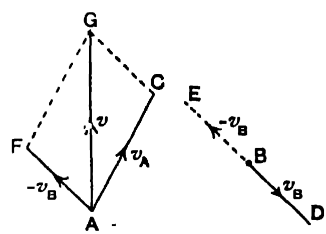


FIG. 42.—Velocity of A relative to B.

a velocity $v_B$ also relative to the paper and represented by BD. To obtain the velocity of B relative to A, stop A by imparting to it a velocity $-v_A$, represented by AE, and equal and opposite to $v_A$; to preserve unaltered the relative conditions, give B also a velocity $-v_A$, represented by BF. A being now at rest relative to the paper, and B having component velocities $v_B$ and $-v_A$, the velocity of B relative to A will be the resultant $v$, obtained by drawing the parallelogram of velocities BDGF.

The velocity of A relative to B may be obtained by a similar process (Fig. 42). B is stopped by imparting to it a velocity $-v_B$, and an equal and like velocity is given to A, represented by AF. A has now components $v_A$ and $-v_B$, which when compounded give a resultant velocity $v$ as the velocity of A relative to B. It will be noted that $v$ in Fig. 41 is equal and opposite to $v$ in Fig. 42.

**Composition and resolution of accelerations.**—Acceleration being a vector quantity, we may say at once that its representation by a straight line, the composition of two or more accelerations in order to find the resultant acceleration, and the resolution of a given acceleration into components along any pair of axes may be carried out in the same manner as for velocity. For example, if a point has an acceleration $a$ represented in magnitude, direction and sense by OA (Fig. 43), the components along two rectangular axes OX and OY will be given by

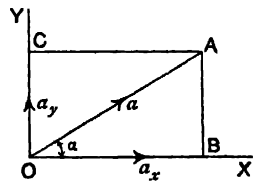


FIG. 43.—Rectangular components of an acceleration.

$$a_x = a \cos \alpha. \quad\quad (1)$$

$$a_y = a \sin \alpha. \quad\quad (2)$$

Also, $$a = \sqrt{a_x^2 + a_y^2}. \quad\quad (3)$$

**Velocity changed in direction.**—In Fig. 44 ($a$) a point has an initial velocity $v_1$, represented by AB, and a final velocity $v_2$, represented by BC.

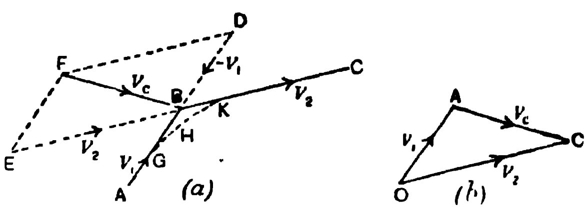


FIG. 44.—Velocity changed from $v_1$ to $v_2$.

To determine what has been the change in velocity during the interval of time we may proceed as follows: Stop the point when it arrives at B by impressing on it a velocity $-v_1$ represented by DB. As the point is at rest now, we may send it off in any direction with any speed. Give it the desired final velocity $v_2$ represented by EB. The resultant change in velocity $v_c$ has components $-v_1$ and $v_2$, and may be found by constructing the parallelogram BDFE, when FB will represent $v_c$.

As will be understood later, it is impossible to have the change in velocity take place instantaneously at B; the direction of motion of any particle or body travelling in the line AB can be changed to another direction BC only by the body travelling along some curve such as that shown dotted at GHK. In this case the total change in velocity during the interval of time in which the particle travels from G to K may be found by the same method and is represented by FB.

A simple alternative construction is shown in Fig. 44 (*b*). A point O is chosen and OA and OC are drawn from it to represent completely $v_1$ and $v_2$ respectively. The total change in velocity $v_c$ is represented by AC, and has a sense indicated by AC, *i.e.* from the end of the initial velocity towards the end of the final velocity. It will be noted that the triangle OAC in Fig. 44 (*b*) is similar and equal to the triangle EFB in Fig. 44 (*a*); hence the truth of the alternative construction is established.

Since the motion along the path GHK has involved a resultant change in velocity $v_c$, it may be asserted that there has been a resultant acceleration having the same direction and sense as $v_c$. This acceleration may be calculated provided the interval of time $t$ is known in which the particle travelled from G to K. Thus:

$$\text{Resultant acceleration} = \frac{v_c}{t}.$$

**Motion in a circular path.**—The case of a point travelling with uniform velocity in the circumference of a circle provides an important application of the above methods. In Fig. 45 (*a*) a point P is travelling in the circumference of a circle of radius $r$ cm., and has a velocity of uniform magnitude $v$ cm. per sec. When the point is at $P_1$, the direction of its velocity will be along the tangent at $P_1$, and is shown by $v_1$. Similarly, when the point is at $P_2$, the velocity has a direction as shown by $v_2$; both $v_1$ and $v_2$ are equal numerically to $v$.

$v_1$ and $v_2$ will meet, if produced, at D; the total change in velocity occurring while the point travels from $P_1$ to $P_2$ may be found by applying the method explained on p. 37. Draw OA to represent $v_1$ (Fig. 45, *b*),

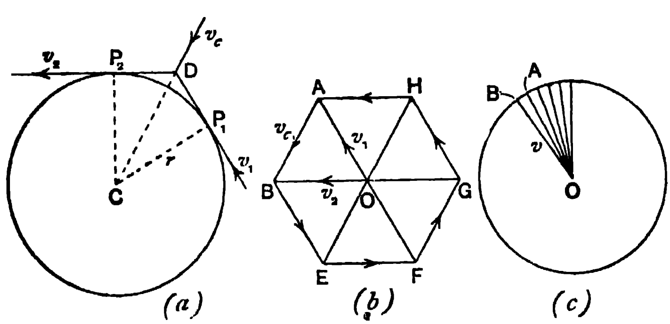


FIG. 45.—Motion in a circle.

also draw OB to represent $v_2$; the total change in velocity will be represented by AB, equal to $v_c$. Apply $v_c$ at D, when it will be evident, from the geometry of the figure, that $v_c$ passes through the centre C of the circle; this fact is independent of the length of the arc $P_1P_2$,

and leads us to assert that **the resultant acceleration of the point is directed constantly towards the centre of the circle.**

In applying this construction there is no limit (other than draughtsmanship) to the smallness of the arc $P_1P_2$. Suppose that this arc is taken very small, then the construction for obtaining the change in velocity becomes OAB (Fig. 45, *c*), in which OA and OB are each made equal to $v$ and AB represents the change in velocity. Repetition of the construction for successive small arcs taken completely round the circle in Fig. 45 (*a*) will give a polygon having a very large number of sides, and this polygon becomes a circle having a radius $v$ when the arcs are taken of indefinite smallness. Thus the total change of velocity while P describes one complete revolution in Fig. 45 (*a*) is given by the circumference of the circle in Fig. 45 (*c*), viz. $2\pi v$ cm. per sec. The interval of time in which this change in velocity takes place is equal to the time taken by P to execute one complete revolution, *i.e.* the time in which P travels a distance of $2\pi r$ along the circumference of the circle in Fig. 45 (*a*). Let $t$ be this time in seconds, then

$$s = vt, \quad \text{(p. 22)}$$

$$2\pi r = vt,$$

or

$$t = \frac{2\pi r}{v} \text{ secs.} \quad \ldots\ldots(1)$$

Also,

$$\text{acceleration} = \frac{\text{change in velocity}}{\text{time interval}};$$

$$\therefore \; a = 2\pi v \div \frac{2\pi r}{v}$$

$$= \frac{v^2}{r} \text{ cm. per sec. per sec.} \quad \ldots\ldots(2)$$

The conclusions are that a point travelling with uniform speed in the circumference of a circle has a constant acceleration directed towards the centre of the circle and given numerically by the above result.

It should be noted that the appropriate British units are $v$ in feet per sec., $r$ in feet and $a$ in feet per sec. per sec. The student may verify this by inserting the dimensions in equation (2).

**Motion in a jet discharged horizontally.**—A jet of water discharged horizontally from a small orifice at O (Fig. 46) provides an interesting example of change in direction of velocity. But for the downward acceleration $g$, which every particle of the water possesses, the jet would continue to travel in the horizontal line OX. Actually

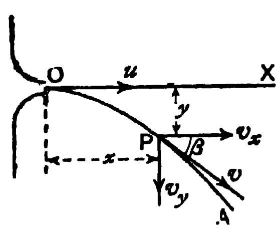

FIG. 46.—Motion in a jet.

it travels in a curved path OPA, and the velocity $v$ of any particle passing through a fixed point P may be taken as compounded of two velocities, viz. $v_x$, which may be assumed to be equal to the initial velocity $u$, and $v_y$, which follows the ordinary laws of falling bodies. These assumptions involve the neglect of effects due to the resistance of the atmospheric air.

Let $t$ be the time taken by a particle in travelling from O to P, and let $x$ and $y$ be the coordinates of P, then

$$x = ut; \quad \therefore \ t = \frac{x}{u}.$$

$$y = \tfrac{1}{2}gt^2 = \tfrac{1}{2}g\frac{x^2}{u^2} = \frac{g}{2u^2}x^2. \quad \ldots\ldots\ldots\ldots \quad (1)$$

Hence, since $g$ and $u$ are constants, $y$ is proportional to $x^2$, and it follows from the geometry of conic sections that the curve of the jet is a parabola.

gain, $$v_x = u, \quad \text{and} \quad v_y = gt = g\frac{x}{u};$$

$$\therefore \ v = \sqrt{v_x{}^2 + v_y{}^2} = \sqrt{u^2 + g^2\frac{x^2}{u^2}}. \quad \ldots\ldots\ldots\ldots \quad (2)$$

Also, $$\cot\beta = \frac{v_x}{v_y} = \frac{u}{gt} = \frac{u^2}{gx}. \quad \ldots\ldots\ldots\ldots \quad (3)$$

For any given value of the initial velocity $u$, the curve of the jet may be plotted from (1); the direction of the tangent to the jet at any time $t$, or at any horizontal distance $x$ from the orifice may be determined from (3), and the velocity at any point in the jet may be found from (2).

**Motion of a particle projected at an angle to the horizontal.**—Referring to Fig. 47, a particle is discharged at O with a velocity $u$ in a line OA

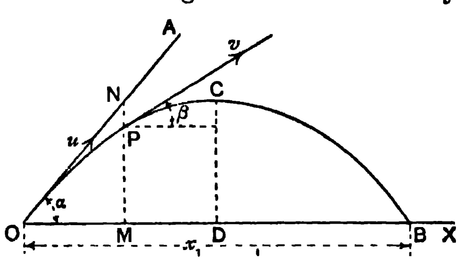


FIG. 47.—Motion of a projectile.

inclined at an angle $\alpha$ to the horizontal. The horizontal and vertical components of $u$ are $u \cos\alpha$ and $u \sin\alpha$ respectively. It may be assumed, neglecting air resistance and any variations in the value of $g$, that $u \cos\alpha$ is the horizontal component of the velocity of the particle at any point

in its flight and is constant, and that $u \sin \alpha$ is affected by the ordinary laws of falling bodies.

Let P be any point on the curved path, or **trajectory**, of the particle ; let $x$ and $y$ be the coordinates of P, and let $t$ be the time taken to travel from O to P. But for the downward acceleration $g$, the particle, after travelling for $t$ seconds, would be found at a point N on OA, vertically over P.

Hence $\quad \text{ON} = ut,$

$$x = \text{ON} \cos \alpha = ut \cos \alpha. \quad \text{.........(1)}$$

Also, $\quad \text{NP} = \tfrac{1}{2}gt^2,$

$$y = \text{MN} - \text{NP} = ut \sin \alpha - \tfrac{1}{2}gt^2. \quad \text{.........(2)}$$

From (1), $\quad t = \dfrac{x}{u \cos \alpha}.$

Substitute in (2), $\quad y = \dfrac{ux \sin \alpha}{u \cos \alpha} - \tfrac{1}{2}g \dfrac{x^2}{u^2 \cos^2 \alpha}$

$$= x \tan \alpha - \frac{g}{2u^2 \cos^2 \alpha} x^2. \quad \text{.........(3)}$$

The form of this relation of $y$ and $x$ indicates that the trajectory is a parabola.

The **horizontal range** is OB (Fig. 47). At B, $y$ is zero ; hence we may obtain the value of $\text{OB} = x_1$ by equating $y$ in (3) to zero :

$$x_1 \tan \alpha - \frac{g}{2u^2 \cos^2 \alpha} x_1^2 = 0 ;$$

$$\therefore \tan \alpha - \frac{g}{2u^2 \cos^2 \alpha} x_1 = 0,$$

$$x_1 = \frac{2u^2 \tan \alpha \cos^2 \alpha}{g} = \frac{2u^2 \sin \alpha \cos \alpha}{g}$$

$$= \frac{u^2 \sin 2\alpha}{g}. \quad \text{.........(4)}$$

The range will be a maximum when $\sin 2\alpha$ is a maximum, *i.e.* when $\sin 2\alpha = 1$ ; $2\alpha$ will then be $90°$ and $\alpha$ will be $45°$. Hence maximum horizontal range will be secured by projection at $45°$ to the horizontal.

In Fig. 47, C is the highest point in the trajectory, and evidently bisects the curve between O and B. The maximum height attained is CD. Let $t_1$ be the total time of flight, then the time taken to reach C from O will be $\tfrac{1}{2}t_1$. Now

$$x_1 = u \cos \alpha \times t_1,$$

or, $$\frac{2u^2 \sin \alpha \cos \alpha}{g} = ut_1 \cos \alpha ;$$

$$\therefore t_1 = \frac{2u \sin \alpha}{g}. \quad \text{.........(5)}$$

and time in which C is reached $= \dfrac{u \sin \alpha}{g}$. .........(5′)

At C, the vertical component of the initial velocity, viz. $u \sin \alpha$, has disappeared; hence, from equation (a), p. 27

$$u^2 \sin^2 \alpha = 2g \times CD,$$

$$CD = \frac{u^2 \sin^2 \alpha}{2g}. \quad \text{......(6)}$$

At P, the velocity $v$ of the particle is inclined at an angle $\beta$ to the horizontal (Fig. 47). Writing $v_x$ and $v_y$ for the horizontal and vertical components of $v$, we have

$$v_x = v \cos \beta = u \cos \alpha,$$

$$v_y = v \sin \beta = u \sin \alpha - gt,$$

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 \cos^2 \alpha + (u \sin \alpha - gt)^2}$$

$$= \sqrt{u^2 (\cos^2 \alpha + \sin^2 \alpha) - 2ugt \sin \alpha + g^2 t^2}$$

$$= \sqrt{u^2 - 2ugt \sin \alpha + g^2 t^2}. \quad \text{......(7)}$$

Also, $$\tan \beta = \frac{v_y}{v_x} = \frac{u \sin \alpha - gt}{u \cos \alpha}. \quad \text{......(8)}$$

## Exercises on Chapter IV

**1.** A point has two component velocities, each equal to 10 cm. per sec. Find, by careful drawing, the resultant velocities when the lines of direction meet at angles of 60°, 120°, 180°, 270°.

**2.** A boat is rowed up a straight reach on a river in a direction making 22° with the bank. If the velocity is 6 feet per second, calculate the component velocities parallel and perpendicular to the bank. In what time will the boat travel a distance of 100 yards, measured parallel to the bank?

**3.** A projectile has component velocities of 1600 feet per second horizontally and 200 feet per second vertically at a certain instant. Calculate the resultant velocity.

**4.** A ship is sailing towards the north-east at 12 miles per hour. A person walks across the deck from port to starboard at 4 feet per second. What is his resultant velocity?

**5.** A piece of coal falls vertically from rest from a height of 9 feet above the floor of a truck travelling at 2 miles per hour. Find the velocity of the coal relative to the truck just before the coal reaches the floor.

**6.** A train has a speed of 30 miles per hour. A drop of rain falls in a vertical plane parallel to the direction of motion of the train. Show in diagrams the direction of motion of the raindrop as seen by an observer in the train, (a) if the raindrop falls vertically with a velocity of 20 feet per second; (b) if the raindrop has, in addition to the velocity given in (a), a component velocity of 5 feet per second in the direction of motion of the train; (c) if the drop has a component of the same magnitude as given in (b) but in a direction opposite to that of the train.

**7.** A person runs after a tramcar travelling at 6 miles per hour. If his velocity is 8 miles per hour in a direction making 30° with the rails, find his velocity relative to the car.

8. A railway coach having ordinary cross-seats is travelling at 8 feet per second. A person about to enter a compartment runs at 10 feet per second. Show in a diagram the direction in which he must run on the platform if his velocity on entering the compartment is to be parallel to the seats; find the magnitude of the latter velocity.

9. A person in a motor car travelling at 30 miles per hour towards the north observes a piece of paper borne by the wind and travelling towards the car apparently from the east with a velocity of 8 feet per second. Find the velocity of the wind.

10. State the parallelogram of velocities. A ship, A, is travelling from south to north with a speed of 20 miles per hour; another ship, B, appears to an observer on A to be travelling from west to east with a velocity of 15 miles per hour. Find the magnitude and direction of B's velocity relative to the earth. L.U.

11. A steamer is travelling northward at the rate of 8 miles an hour in a current flowing westward at the rate of 3 miles an hour. Indicate in a diagram the direction in which the steamer is heading, and find the rate at which it is steaming. If the wind is blowing at the rate of 3 miles an hour from the east, indicate in your diagram the direction in which a small flag at the masthead is pointing. L.U.

12. An aeroplane is travelling towards the north-west relative to the earth at 180 miles per hour, and the wind is blowing at 40 miles per hour towards the north. Suppose the wind were to cease suddenly, find the velocity of the aeroplane in magnitude and direction relative to the earth.

13. Two railway tracks $Ox$ and $Oy$ enclose an angle of 60°. A train moves along $Ox$ with uniform velocity of 60 miles an hour, while a second train moves along $Oy$ with equal speed, passing through O two minutes after the first. Find the velocity of the second train relatively to the first, and indicate in a diagram the shortest distance between the trains. L.U.

14. A cyclist rides at 10 miles an hour due north, and the wind (which is blowing at 6 miles an hour from a point between N. and E.) appears to the cyclist to come from a point 15° to the east of north; find graphically or by calculation the true direction of the wind, and the direction in which the wind will appear to meet him on his return, if he rides at the same speed.

15. Two ships are steaming along straight courses with such constant velocities that they will collide unless their velocities are altered. Show that to an observer on either ship the other appears to be always moving directly towards him. L.U.

16. Explain what is meant by the velocity of one moving particle relative to another moving particle, and show how to determine it. To a ship sailing E. at 15 knots another ship whose speed is 12 knots appears to be sailing N.W. Show that there are two directions in which the latter may be moving. Find these directions, graphically or otherwise, and find the relative velocity in each case. L.U.

17. A railway coach at a certain instant has a velocity of 10 metres per second towards the north. Twenty seconds afterwards the velocity is found to be 15 metres per second towards the north-west. Find the change in velocity and the average value of the acceleration.

18. A piece of tube is bent near the middle so that the straight portions include an angle of 30°. If water flows through the tube with uniform velocity of 4 feet per second, find the total change in velocity in passing round the bend.

19. A billiard ball travelling at 3 feet per second strikes the cushion and moves thereafter in a line making 60° with the original direction of motion and with a velocity of $2\frac{1}{2}$ feet per second. Find the change in velocity.

20. A point travels in the circumference of a circle 40 cm. in diameter with a uniform velocity of 120 cm. per second. Find the acceleration towards the centre of the circle.

21. A railway coach has a speed of 60 miles per hour and travels round a curve having a radius of 1200 feet. Find the acceleration towards the centre of the curve.

22. A jet of water issues from a small hole in the vertical side of a tank with a horizontal velocity of 8 feet per second. Find the resultant velocity of a particle in the jet 2 seconds after it has issued from the orifice.

23. In Question 22, find the position of a particle in the jet at intervals of 0·1, 0·2, 0·3, 0·4 and 0·5 second after issue. From the information so obtained plot a graph showing the shape of the jet. Take $g=32$ ft. per sec. per sec.

24. A bullet is projected with a velocity of 1200 feet per sec. horizontally from a gun which is 25 feet above the ground. Find the horizontal distance from the gun at which the bullet strikes the ground, and also the angle its direction of motion then makes with the horizontal.

25. A projectile is fired with a velocity of 2200 feet per second. Find the horizontal range, time of flight and greatest height attained when the angles of elevation are respectively 30°, 40°, 45°, 50° and 60°. Neglect atmospheric effects. Take $g=32$ ft. per sec. per sec.

26. A gun capable of firing a projectile with a velocity of 2000 feet per second is placed at a horizontal distance of 400 feet from the foot of a vertical cliff 200 feet high. Find the angle of elevation of the gun in order that the projectile may just clear the edge of the cliff. Neglect atmospheric effects.

27. A ball is projected from a point 7 feet high with a velocity of 50 feet per second. At what angle to the horizontal must it be projected in order just to clear the top of a net 3·5 feet high at a horizontal distance of 30 feet from the point of projection? Neglect atmospheric effects.

28. A heavy particle is projected with a velocity $v$ in a direction making an angle $\theta$ with the horizon. Form the equations determining its position and velocity at any subsequent instant of time. Drops of water are thrown tangentially off the horizontal rim of a rotating umbrella. The rim is 3 feet in diameter, and is held 4 feet above the ground, and makes 14 revolutions in 33 seconds. Show that the drops of water will meet the ground on a circle 5 feet in diameter.

29. Show how to obtain the relative velocity of two points whose motion is given.

A ship A is moving with uniform velocity. To a passenger on another ship B going due east at 14 knots she appears to move due north, but when the speed of B is reduced to 8 knots she appears to be moving 30° east of north. Show that the speed of A is $4\sqrt{19}$ knots. C.W.B., H.C.

**30.** Prove that, if atmospheric resistances are neglected, the path of a projectile is a parabola.

An aeroplane is flying at a height of 3000 ft. in a straight horizontal course at a speed of 150 miles per hour, the direction of the course being such as to carry it vertically over a fort, on which the pilot has to drop a bomb. Find the angle between the vertical and the straight line joining the aeroplane to the fort at the moment when the bomb should be released.

J.M.B., H.S.C.

# CHAPTER V

## ANGULAR VELOCITY AND ACCELERATION

**Angular velocity.**—Let one point in a straight line be fixed, and let the line revolve about this point in a fixed plane, say that of the paper. The rate of describing angles is termed the **angular velocity** of the line. Angular velocity may be measured in revolutions per minute or per second ; for many purposes it is more convenient to measure angular velocity in radians per second. The symbol $\omega$ is used to denote the latter.

Since there are $2\pi$ radians in a complete revolution, the connection between $\omega$ and the revolutions per minute, N, is

$$\omega = \frac{\mathsf{N}}{60} 2\pi = \frac{\pi \mathsf{N}}{30} \text{ radians per second.}$$

In uniform angular velocity, equal angles are described in equal intervals of time ; should this condition not be complied with the angular velocity varies, and the revolving line is said to have **angular acceleration.**

Angular velocity may be described as being **clockwise** or **anticlockwise,** according as the line appears to the observer to rotate in the same, or in the opposite direction to that of the hands of a clock. The student will note that, if there are two observers, one on each side of the plane of rotation, the angular velocity will appear to be clockwise to one observer and anticlockwise to the other.

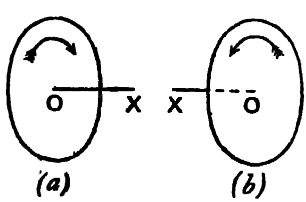

FIG. 48.—Representation of angular velocity.

A given angular velocity may be represented by drawing a line perpendicular to the plane in which the body is revolving. The length of the line represents the angular velocity to a chosen scale, and the line is drawn on one or the other side of the plane of revolution depending on the sense of rotation. Thus, in Fig. 48 (*a*), a person situated on the right-hand side of the revolving disc observes that the angular velocity is clockwise and draws OX perpendicular to the plane of the disc and on his side of the disc. In Fig. 48 (*b*), the angular velocity appears to the person to be anticlockwise, and OX is drawn on the opposite side of the disc. The student should verify by trial that two persons on opposite sides of a revolving disc will agree in placing OX on the same side of the disc.

**Relation of linear and angular velocity.**—Let OA (Fig. 49) revolve about O with uniform angular velocity. At any instant the point A has a linear velocity $v$ in the direction at right angles to OA. Let $r$ be the radius of the circle which A describes. The length of the arc

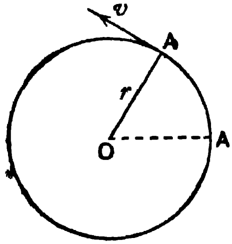


FIG. 49.—Relation of angular and linear velocities.

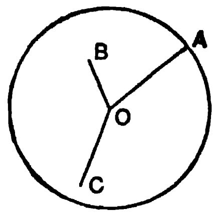


FIG. 50.—All radial lines have the same angular velocity.

described by A in one second is $v$, and the angle subtended at the centre of the circle by this arc will be $v/r$ radians, the same unit of length being used in stating both $v$ and $r$. Hence OA turns through $v/r$ radians in one second, and the angular velocity is

$$\omega = \frac{v}{r} \text{ radians per second.} \qquad .(1)$$

In Fig. 50 a wheel rotates in the plane of the paper about an axis at O perpendicular to this plane. It is evident that the radii drawn to any fixed points, OA, OB, OC, etc., all possess the same angular velocity. Hence the angular velocity of a rotating body may be calculated by dividing the linear velocity of any point in the body by the radius drawn from that point to the axis of rotation.

**Angular acceleration.**—Angular acceleration is defined as the rate of change of angular velocity, and may be calculated by dividing the change in angular velocity by the time taken. Thus, if a revolving line changes its angular velocity from $\omega_1$ to $\omega_2$ radians per second in $t$ seconds, and if the change has been effected at a uniform rate, then

$$\text{Angular acceleration} = \phi = \frac{\omega_2 - \omega_1}{t} \text{ radians per sec. per sec.} \qquad ...(2)$$

In Fig. 51 a line rotates about O in the plane of the paper with varying angular velocity. When passing through OA its angular velocity is $\omega_1$, and the angular velocity increases at a uniform rate and is $\omega_2$ when passing through OB. Let the time taken to travel from OA to OB be $t$ seconds, then

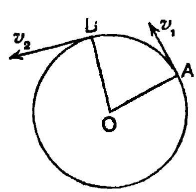


FIG. 51.—Angular acceleration.

$$\text{Angular acceleration} = \phi = \frac{\omega_2 - \omega_1}{t}.$$

Let the linear velocities of A and B be $v_1$ and $v_2$ respectively, and let $r$ be the radius of the circle, then

$$\omega_1 = \frac{v_1}{r}, \quad \text{and} \quad \omega_2 = \frac{v_2}{r};$$

$$\therefore \ \phi = \frac{v_2 - v_1}{rt}.$$

Now $(v_2 - v_1)/t$ is the tangential acceleration $a$ of the point A in travelling from A to B, hence

$$\phi = \frac{a}{r}. \qquad \text{......................................(3)}$$

It will be noted that this rule corresponds with that for deriving angular velocity from linear velocity.

All radii of a revolving body possess the same angular acceleration, hence the angular acceleration may be calculated by dividing the tangential acceleration of any point in the body by the radius drawn to the point from the axis of rotation.

**Equations of angular motion.**—Equations for angular motion may be derived in the manner adopted in Chapter III in finding equations for rectilinear motion.

Let a line revolve with uniform angular velocity $\omega$ radians per second, and let $\alpha$ be the angle described in $t$ seconds. Then

$$\alpha = \omega t \text{ radians.} \qquad \text{.................................(1)}$$

Let a line start to revolve from rest with an angular acceleration $\phi$ radians per second per second, the angular velocity $\omega$ at the end of $t$ seconds is given by

$$\omega = \phi t \text{ radians per second.} \qquad \text{..........................(2)}$$

The average angular velocity is

$$\omega_a = \frac{0 + \omega}{2} = \tfrac{1}{2}\omega,$$

and $$\alpha = \omega_a t = \tfrac{1}{2}\omega t \text{ radians.} \qquad \text{..............................(3)}$$

Substituting for $\omega$ from (2) gives

$$\alpha = \tfrac{1}{2}\phi t \times t = \tfrac{1}{2}\phi t^2 \text{ radians.} \qquad \text{..........................(4)}$$

Again, from (2), $\quad t = \dfrac{\omega}{\phi}; \quad \therefore \ t^2 = \dfrac{\omega^2}{\phi^2}.$

Substituting this value in (4), we have

$$\alpha = \tfrac{1}{2}\phi \frac{\omega^2}{\phi^2} = \frac{\omega^2}{2\phi};$$

$$\therefore \ \omega^2 = 2\phi\alpha. \qquad \text{................................................(5)}$$

The analogy of these equations with those for rectilinear motion is apparent. Equations for a line having an initial angular velocity $\omega_1$ and an angular acceleration $\phi$ may be obtained in a similar manner. The equations are as follows :

$$\omega_2 - \omega_1 = \phi t \text{ radians per sec.} \quad \text{.........(6)}$$

$$\alpha = \left(\frac{\omega_2 + \omega_1}{2}\right) t \text{ radians.} \quad \text{.........(7)}$$

$$\alpha = \omega_1 t + \tfrac{1}{2}\phi t^2 \text{ radians.} \quad \text{.........(8)}$$

$$\omega_2^2 - \omega_1^2 = 2\phi\alpha. \quad \text{.........(9)}$$

EXAMPLE 1.—A wheel starts from rest and acquires a speed of 300 revolutions per minute in 40 seconds. Find the angular acceleration. How many revolutions did the wheel describe during the 40 seconds?

$$\omega = \frac{300}{60} \cdot 2\pi = 10\pi = 31 \cdot 41 \text{ radians per sec.}$$

$$\phi = \frac{\omega}{t} = \frac{31 \cdot 41}{40} = \underline{0 \cdot 785} \text{ radian per sec. per sec.}$$

$$\text{Average angular velocity} = \frac{300}{2} = 150 \text{ revs. per min.}$$

$$= \frac{150}{60} = 2 \cdot 5 \text{ revs. per sec.}$$

$$\therefore \text{Revolutions described} = 2 \cdot 5 \times 40 = \underline{100}.$$

EXAMPLE 2.—The driving wheel of a locomotive is 6 feet in diameter. Assuming that there is no slipping between the wheel and the rail, what is the angular velocity of the wheel when the engine is running at 60 miles per hour?

$$\text{Velocity of locomotive} = \frac{5280 \times 60}{60 \times 60} = 88 \text{ ft. per sec.}$$

As the distance travelled in one second is 88 feet, we may find the revolutions of the wheel per second by imagining 88 feet of rail to be wrapped round the circumference of the wheel.

$$\text{Number of turns of rail} = \frac{88}{\pi d} = \frac{88}{6\pi}.$$

$$\therefore \text{ Revolutions per second} = \frac{88 \times 7}{22 \times 6} = 4 \cdot 67 \text{ ;}$$

$$\therefore \omega = 4 \cdot 67 \times 2\pi$$

$$= \underline{29 \cdot 33} \text{ radians per sec.}$$

**Transmission of motion of rotation.**—In workshops many machines are driven by means of belts. A pulley is fixed to each shaft, and a belt is stretched round the pulleys as shown in Fig. 52. If it is intended that both shafts should rotate in the same direction, the belt is open

as in Fig. 52. Crossing the belt as shown in Fig. 53 enables one shaft to drive the other in the contrary direction. Neglecting any slipping between the belt and the pulleys, it is evident that the linear velocities

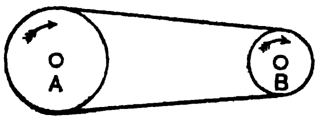


FIG. 52.—Open belt.

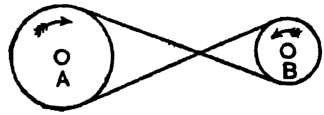


FIG. 53.—Crossed belt.

of points on the circumferences of both pulleys are equal to the linear velocity of the belt. Let $V$ be this velocity, and let $R_A$ and $R_B$ be the radii of the pulleys, then

$$\text{Angular velocity of } A = \omega_A = \frac{V}{R_A}.$$

$$\text{Angular velocity of } B = \omega_B = \frac{V}{R_B}.$$

$$\therefore \frac{\omega_A}{\omega_B} = \frac{R_B}{R_A}.$$

Thus the angular velocities of the pulleys are inversely proportional to the radii, or the diameters of the pulleys.

The arrangement shown in Fig. 54 enables a larger angular velocity ratio to be obtained. A drives B, and another pulley C, fixed to the

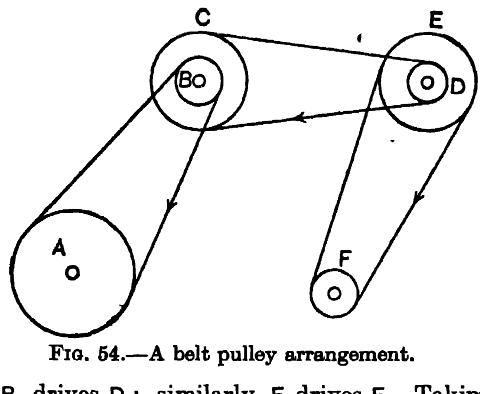


FIG. 54.—A belt pulley arrangement.

same shaft as B, drives D; similarly, E drives F. Taking the pulleys in pairs, we have

$$\frac{\omega_A}{\omega_B} = \frac{R_B}{R_A}; \quad \frac{\omega_C}{\omega_D} = \frac{R_D}{R_C}; \quad \frac{\omega_E}{\omega_F} = \frac{R_F}{R_E}.$$

Also $\qquad \omega_B = \omega_C, \quad \text{and} \quad \omega_D = \omega_E.$

Hence $$\frac{\omega_A \times \omega_C \times \omega_E}{\omega_B \times \omega_D \times \omega_F} = \frac{R_B \times R_D \times R_F}{R_A \times R_C \times R_E},$$

or $$\frac{\omega_A}{\omega_F} = \frac{R_B \times R_D \times R_F}{R_A \times R_C \times R_E}.$$

Hence the rule: the angular velocity ratio of the first and last pulleys is given by the product of the radii, or diameters, of the driven pulleys divided by the product of the radii, or diameters, of the driving pulleys.

**Toothed wheels** (Fig. 55) are used in cases where there must be no slipping. The teeth may be imagined to be formed on two cylinders shown dotted. It is clear that the linear velocities of points on the circumferences of the cylinders are equal, and therefore we have the same rule as for a pair of belt pulleys, viz.

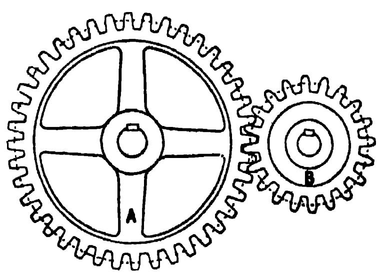


Fig. 55.—Toothed wheels in gear.

$$\frac{\omega_A}{\omega_B}=\frac{R_B}{R_A}.$$

Further, the numbers of teeth, $n_A$ and $n_B$, are proportional to the circumferences and therefore to the radii of the cylinders. Hence

$$\frac{\omega_A}{\omega_B}=\frac{n_B}{n_A}.$$

The wheels shown in Fig. 55 revolve in opposite directions. If angular velocities of the same sense be required, an **idle wheel C** is interposed (Fig. 56). The linear velocities of the circumferences of all three cylinders are still equal; hence, as before,

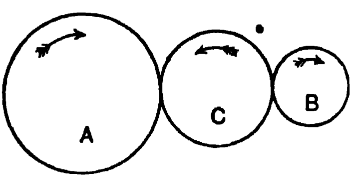


Fig. 56.—Use of an idle wheel.

$$\frac{\omega_A}{\omega_B}=\frac{n_B}{n_A}.$$

A **train of wheels**, such as is used in clocks and other devices, is shown in Fig. 57. Taking the wheels in pairs, we have

$$\frac{\omega_A}{\omega_B}=\frac{n_B}{n_A};\quad \frac{\omega_C}{\omega_D}=\frac{n_D}{n_C};\quad \frac{\omega_E}{\omega_F}=\frac{n_F}{n_E}.$$

Also, $\omega_B=\omega_C$, and $\omega_D=\omega_E$.

Hence

$$\frac{\omega_A\times\omega_C\times\omega_E}{\omega_B\times\omega_D\times\omega_F}=\frac{\omega_A}{\omega_F}=\frac{n_B\times n_D\times n_F}{n_A\times n_C\times n_E}.$$

It will be noticed that this result is similar to that obtained for the train of belts shown in Fig. 54.

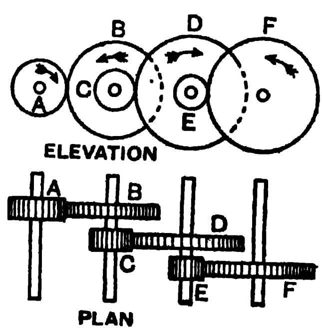


Fig. 57.—Train of wheels.

**Chain drives** are sometimes used instead of belts in order to avoid slipping. An

ordinary bicycle provides an example. The circumferences of the toothed chain wheels, taken at the centres of the links of the chain, have the same linear velocity as the chain, hence we have the same rule as in the case of two belt pulleys, viz.

$$\frac{\omega_A}{\omega_B}=\frac{R_B}{R_A}.$$

Further, the numbers of teeth on the wheels are proportional to the circumferences, and therefore to the radii of the wheels ; hence

$$\frac{\omega_A}{\omega_B}=\frac{n_B}{n_A}.$$

In early bicycles the driving was accomplished by means of cranks fixed to the axle of the front wheel ; thus one revolution of the crank gave one revolution to the wheel, and moved the bicycle through a distance equal to the circumference of the wheel. When the statement is made that the gear of a modern bicycle is so much, say 70, it is meant that for one revolution of the cranks the bicycle will travel a distance equal to that which would be covered by an old-fashioned machine having a driving wheel 70 inches in diameter. Let $d$ be the diameter of the back wheel of the safety bicycle, and let $n_A$ and $n_B$ be the numbers of teeth on the crank chain wheel and the small chain wheel respectively, then

$$\text{Gear}=D=\frac{n_A}{n_B}d.$$

EXAMPLE.—**Varying angular velocity.** In Fig. 58 a point travels with uniform velocity $v$ in the straight line $XP_1$. The angular velocity of the radius vector $OP_1$ drawn from any fixed point O to the moving point at any instant may be determined thus : Consider two successive positions of the point, $P_1$ and $P_2$, and let these be close together. Join $OP_1, OP_2$, and draw $P_1K$ perpendicular to $OP_2$. Let the angle $P_1OX$ be $\alpha$, and let the angle $P_1OP_2$ be called $\delta\alpha$. If $\delta t$ is the time in which the point travels from $P_1$ to $P_2$, the radius vector describes the angle $\delta\alpha$ in the same time, and the angular velocity is given by

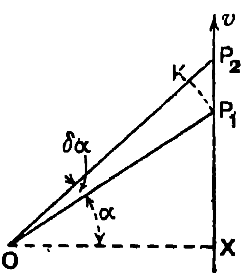


FIG. 58.—Varying angular velocity.

$$\omega=\frac{\delta\alpha}{\delta t}.$$

In the similar triangles $P_1OX$, $P_1KP_2$, the angle $KP_1P_2=P_1OX=\alpha$.

Now $$\delta\alpha=\frac{P_1K}{OP_1}=\frac{P_1P_2 \,.\, \cos\alpha}{OP_1}=\frac{P_1P_2 \,.\, \cos\alpha \,.\, \sin\alpha}{P_1X}.$$

Also $$P_1P_2=v \,.\, \delta t ;$$

$$\therefore\ \delta\alpha=\frac{v \,.\, \delta t \,.\, \cos\alpha \,.\, \sin\alpha}{P_1X} ;$$

$$\therefore\ \omega=\frac{\delta\alpha}{\delta t}=\frac{v \,.\, \sin\alpha \,.\, \cos\alpha}{P_1X}. \quad \text{......(1)}$$

This expression gives the angular velocity of the radius vector in terms of the distance of the point from X. If $P_1X$ be zero, the point is passing through X, and $\alpha$ is zero. The expression for $\omega$ then takes the form 0/0. To determine the value, let the point be taken very close to X, when $\sin\alpha = \frac{P_1X}{OP_1}$ and $\cos\alpha = 1$. Inserting these values gives

$$\omega_0 = \frac{v \cdot P_1X}{P_1X \cdot OP_1} = \frac{v}{OP_1}$$

$$= \frac{v}{OX}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

a result which complies with equation (1), p. 47.

**Relative angular velocity.**—In Fig. 59 a point A at a certain instant has a velocity $v_A$ and another point B has a velocity $v_B$ at the same instant. Stop A by impressing on it a velocity $-v_A$ and impress the same velocity $-v_A$ on B. Find the resultant velocity of B by means of the parallelogram B*acb*; this will be $v$. Instead of the given conditions of motion we now have the following equivalent conditions: A point A is at rest, and another point B is travelling along a straight line B*c* with uniform velocity $v$ and has reached B at a certain instant. Draw AX perpendicular to B*c*, producing the latter if necessary. Let the angle BAX be called $\alpha$, then, from equation (1) (p. 52), we have

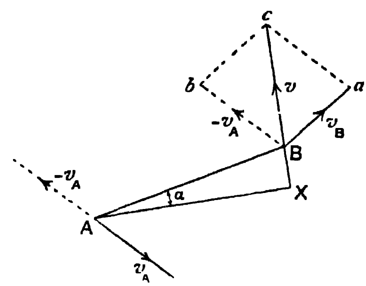


FIG. 59.—Relative angular velocity.

$$\text{Relative angular velocity of B with respect to A} = \frac{v \cdot \sin\alpha \cdot \cos\alpha}{BX}.$$

With the velocities as given in Fig. 59, this relative angular velocity is counterclockwise. The angular velocity of A relative to B may be found in a similar manner, stopping B by applying $-v_B$ to B. The student should draw the diagram for this case for himself, and should verify that the angular velocity of A relative to B is equal to that of B relative to A, and has the same sense of rotation.

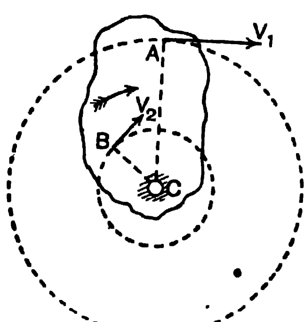


FIG. 60.—Velocities of points in a rotating body.

**Velocity of any point in a rotating body.**—In Fig. 60 is shown a body rotating with uniform angular velocity $\omega$ about an axis at C which is perpendicular to the plane of the paper. At any instant the direction of the velocity of any point, such as A or B, is perpendicular to the

radius. Suppose that the velocity of A is given, equal to $V_1$ say, the velocity $V_2$ of B may be calculated as follows:

$$\omega = \frac{V_1}{AC} = \frac{V_2}{BC}; \quad \therefore \frac{V_1}{V_2} = \frac{AC}{BC},$$

a result which shows that the velocity of any point is proportional to its distance from the axis of rotation.

**Instantaneous centre of rotation.**—Let a rod AB (Fig. 61) be moving in such a manner that at a given instant A has a velocity $V_A$ and B has a velocity $V_B$ in the directions shown. The direction of $V_A$ will not be altered if we imagine that A is rotating for an instant about any centre in a line AI drawn perpendicular to $V_A$. Similarly, $V_B$ will not be altered in direction if we imagine B to be rotating for an instant about any centre in BI which is perpendicular to $V_B$. These perpendiculars intersect at I, and we may consider that both A and B are rotating for an instant about I without thereby changing the directions of their velocities. I is called the **instantaneous centre of rotation**. It is evident that, if two points in the rod rotate for an instant about I, every point in the rod is rotating about I at the same instant.

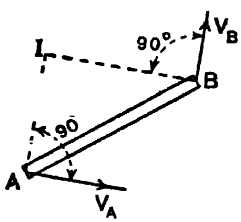

FIG. 61.—Instantaneous centre.

If $V_A$ is known, we may calculate $V_B$ from the relation given on p. 53, viz.

$$\frac{V_A}{V_B} = \frac{IA}{IB}.$$

EXAMPLE.—In Fig. 62 is shown a slider-crank mechanism (p. 20) in which the crank BC rotates with uniform angular velocity in the plane of the paper about an axis at C. The rod AB is jointed to BC at B, and its end A is constrained to move in the line AC. Knowing the velocity $V_B$ of B at any instant, the velocity of A may be found by application of the instantaneous centre method. Draw AI perpendicular to AC; then A may be imagined for an instant to be rotating about any centre in AI. Draw BI perpendicular to $V_B$, *i.e.* produce CB; B may be imagined to rotate for an instant about any centre in BI. Hence I is the instantaneous centre for the rod AB, and we have

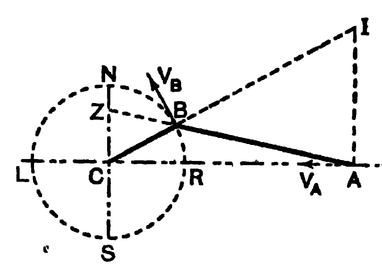

FIG. 62.—Instantaneous centre of AB.

$$\frac{V_A}{V_B} = \frac{IA}{IB}.$$

In some positions of the mechanism, I will fall at a large distance from AC; when BC is at 90° to AC, I lies at infinity. A simple modification brings the whole construction required within the boundary of a piece of drawing paper of moderate size.

Draw NCS through C at 90° to AC; produce AB (if necessary) to cut NCS in Z. It is evident that the triangles IAB and CZB are similar; hence

$$\frac{IA}{IB}=\frac{CZ}{CB};$$

$$\therefore \frac{V_A}{V_B}=\frac{CZ}{CB}=\frac{1}{R}.CZ,$$

where R is the length of the rod BC. Since R and $V_B$ are both constants, it follows that $V_A$ is proportional to CZ.

**A rolling wheel.**—In Fig. 63 (*a*) is shown a wheel rolling along a road without slipping. It is evident that the velocity of the centre of

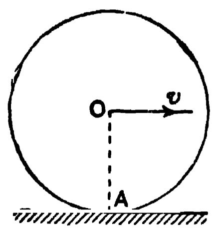


FIG. 63 (*a*).—Angular velocity of a rolling wheel.

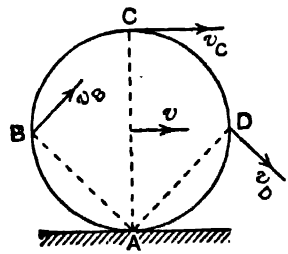


FIG. 63 (*b*).—Velocities of points in a rolling wheel.

the wheel, O, is equal to the velocity $v$ of the vehicle to which the wheel is attached. Further, if there is no slipping, then the point A in the wheel rim, being in contact with the ground for an instant, is at rest, and is therefore the instantaneous centre of rotation. Hence the angular velocity of the wheel is $v/OA$, a result agreeing with that found on p. 49 by another method.

Every point in the wheel is rotating for an instant about A; hence the velocity of any point may be found. Thus the velocity of C (Fig. 63 (*b*)) is at 90° to AC and is given by

$$\frac{v_c}{v}=\frac{AC}{AO}=2;$$

$$\therefore v_c=2v.$$

The velocities of B and D (situated on the horizontal line passing through O) are perpendicular respectively to AB and AD, and are given by

$$\frac{v_B}{v}=\frac{AB}{AO}=\sqrt{2};$$

$$v_B=v\sqrt{2}.$$

Similarly, $$v_D=v\sqrt{2}.$$

## Exercises on Chapter V

**1.** A wheel revolves 90 times per minute. Find its angular velocity in radians per second.

**2.** What is the angular velocity in radians per second of the second hand of a watch?

**3.** Find the revolutions per minute described by a wheel which has an angular velocity of 30 radians per second.

**4.** A revolving wheel changes speed from 50 to 49 radians per second. State the change in revolutions per minute.

**5.** A point on the rim of a wheel 8 feet in diameter has a linear velocity of 48 feet per second. Find the angular velocity of the wheel.

**6.** A wheel starts from rest and acquires a speed of 200 revolutions per minute in 24 seconds. Find the angular acceleration.

**7.** Find the angular acceleration of a wheel which undergoes a change in angular velocity from 50 to 48 radians per second in 0·5 second.

**8.** A point on the rim of a revolving wheel 8 feet in diameter has a velocity in the direction of the tangent of 80 feet per second. Five seconds afterwards the same point has a tangential velocity of 60 feet per second. Find the angular acceleration of the wheel.

**9.** A wheel starts from rest with an angular acceleration of 0·2 radian per second per second. In what time will it acquire a speed of 150 revolutions per minute? How many revolutions will it make during this interval of time?

**10.** Find the angle turned through by a wheel which starts from rest and acquires an angular velocity of 30 radians per second with a uniform acceleration of 0·6 radian per second per second.

**11.** A wheel changes speed from 140 to 150 revolutions per minute and describes 40 revolutions while doing so. Find the angular acceleration.

**12.** Find the angular velocity of a bicycle wheel 28 inches diameter when the bicycle is travelling at 12 miles per hour. How many revolutions will the wheel describe in travelling a distance of one mile?

**13.** A shaft A drives another shaft B by means of pulleys and a belt. If the pulley on A is 24 inches in diameter and runs at 200 revolutions per minute, find the diameter of the pulley on B in order that it may have a speed of 150 revolutions per minute.

**14.** A small motor has a pulley 2 inches in diameter and runs at 1200 revolutions per minute. A shaft having a pulley 12 inches in diameter is driven by a belt passing round the motor pulley. On the same shaft is another pulley 3 inches in diameter connected by a belt to a pulley 10 inches in diameter and fixed to the shaft of an experimental model. Find the speed in revolutions per minute of the model shaft.

**15.** The driving wheel of a bicycle is 28 inches in diameter, and has a sprocket wheel having 18 teeth. The chain wheel on the crank axle has 46 teeth. What is the "gear" of the bicycle? How many revolutions must each crank make in travelling a distance of one mile?

**16.** A wheel A having 20 teeth drives another wheel B having 54 teeth. If A runs at 110 revolutions per minute, find the speed of revolution of B. Show how A and B could be run at the same speeds as before, but both in the same direction of rotation.

**17.** In winding a watch 3·5 complete turns are given to the spring case; this serves to keep the watch going for 28 hours. What is the ratio of the angular velocities of the spring case and the minute hand during the ordinary working of the watch?

**18.** The minute hand of a watch is connected to the hour hand by a train of wheels. A wheel A on the minute hand spindle has 12 teeth and drives a wheel having 48 teeth; on the same spindle as the latter wheel is another having 8 teeth, and this wheel drives a wheel having N teeth on the hour-hand spindle. Find N.

**19.** Explain how angular velocity is measured. A point P moves with uniform velocity $v$ along a straight line. ON is drawn perpendicular to this line, O being a fixed point. Express the angular velocity of P about O in terms of the distance OP. L.U.

**20.** If two particles describe the circle of radius $a$, in the same sense and with the same speed $u$, show that the relative angular velocity of each with respect to the other is $u/a$. L.U.

**21.** A rod OA is pivoted to a fixed point at O, and is freely jointed at A to a second rod AB; the end B is constrained to move in a straight groove passing through O. If the rod OA rotates about O with uniform angular velocity $\omega$, show that the velocity of B at any instant is

$$\text{OA}\,(\sin\theta + \cos\theta\tan\phi)\,\omega,$$

where $\theta$ and $\phi$ are the acute angles made by OB with OA and AB at the instant. L.U.

**22.** Find the velocity at any point on the rim of a wheel rolling with uniform velocity $v$ along a horizontal plane without sliding. Show that each point of the wheel moves as though it were revolving about the point of contact of the wheel and the ground at the instant.

**23.** A disc moves in its plane in such a way that a point O on it is moving at any instant with velocity components $u$, $v$ in fixed perpendicular directions whilst the disc is rotating about O with angular velocity $\omega$. Prove that, referred to axes in the fixed directions through O, the point whose coordinates are $-v/\omega$, $u/\omega$ is instantaneously at rest and the disc is moving as though it were rotating about this point with angular velocity $\omega$.

C.W.B., H.C.

# CHAPTER VI

## NEWTON'S LAWS OF MOTION

**Newton's first law of motion.**—The whole science of dynamics is based on three fundamental laws formulated by Newton. The **first law** is as follows :

**Every body continues in its state of rest or of uniform motion in a straight line except in so far as it is compelled by forces to change that state.**

The term **inertia** is given to the tendency of a body to preserve its state of rest, or of constant rectilinear velocity. The first law expresses the results of experience. A train at rest on a level track will not move until the locomotive applies a tractive force. If the train is travelling with constant speed, the engine exerts a pull sufficient merely to overcome the frictional resistances, and must exert a considerably greater pull while the speed is being increased. If steam be shut off, the frictional resistances gradually reduce the speed, and if the brakes be applied, the increased frictional forces bring the train to rest quickly. Thus a force having the same sense and direction as the velocity must be applied in order to obtain an increase in velocity, and a force having the opposite sense if the velocity has to be diminished.

A person standing on the top of a tramcar may experience the effects of inertia in his body ; should the driver apply the brakes suddenly, the passenger will be shot forward. If the driver starts rapidly, the passenger will be left behind as it were. Should the car reach a curve on the track it will follow the track, and the passenger's body will endeavour to proceed rectilinearly, and will incline towards the outside of the curve.

To cause any body to travel in a curved path requires the application of a force in a direction transverse to that of the path.

We now proceed to discuss some principles leading to Newton's second law of motion.

**Relation of force, mass and acceleration.**—**All bodies falling freely at the same place will have equal accelerations.** This may be confirmed by experiment. Two stones released simultaneously from the same height will reach the ground at the same instant. If a piece of paper be substituted for one of the stones, the paper will take a longer time to fall ; this effect is owing to the resistance of the air, and may be got rid of partially by crumpling the paper into a ball, when it will be found that both stone and paper fall together.

Since the weights of two bodies are proportional to the masses, and since both bodies fall freely with equal accelerations, it follows that **the forces required are proportional to the masses if equal accelerations are to be imparted to a number of bodies.**

A laboratory experiment (p. 62) may be devised to illustrate another law, viz. **the force which must be applied to a body of given mass is proportional to the acceleration required.**

Combining these statements leads to the general law: **The force required is proportional jointly to the mass and the acceleration, and is therefore measured by the product of mass and acceleration.** The acceleration takes place in the same direction and sense as the applied force.

Let $F =$ the force applied to the body by the external agent.
$m =$ the mass of the body.
$a =$ the acceleration.

Then $$F = ma.$$

**Absolute units of force.**—Convenient absolute units of force (p. 6) may be derived from the above result. Take $m$ to be the unit of mass and $a$ to be the unit of acceleration in any given system; then $F$ becomes unity and may be accepted as the absolute unit of force for the system. The C.G.S. and British absolute units of force have been defined on p. 6, and are restated here in a slightly different form:

A force of one **dyne** applied to a gram mass produces an acceleration of one centimetre per second per second.

A force of one **poundal** applied to a pound mass produces an acceleration of one foot per second per second.

The dimensions of force may be deduced from the above equation by substitution.

Thus: $$F = ma = m\frac{l}{t^2}, \quad \text{or} \quad mlt^{-2}.$$

**Relation of absolute and gravitational units of force.**—Since a body of mass $m$ falls freely under the influence of its weight $W$ and has an acceleration $g$, it follows that the weight of a body, expressed in absolute units of force, is given by:

$$W = mg.$$

A force of one lb. weight, acting on a mass of one pound falling freely, produces an acceleration of $g$ feet per second per second. A force of $g$ poundals would produce the same acceleration; hence one lb. weight is equivalent to $g$ poundals. Similarly, one gram weight is equivalent to $g$ dynes. In interpreting these statements it will be understood that $g$ must be in feet, or centimetres, per second per second according to the system employed.

**To convert from gravitational to absolute force units, multiply by g.**

**Newton's second law of motion.**—Suppose a body of mass $m$ to be at rest in the initial position A (Fig. 64). If a force F be applied, a constant acceleration $a$ will occur; let this continue during a time interval $t$ seconds, and let the body travel from A to B during this interval, the velocity being $v$ on reaching B. We have:

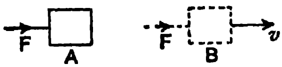

FIG. 64.—Relation of force and momentum generated.

$$F = ma.$$

Also, $$v = at \text{ (p. 25)}, \quad \text{or} \quad a = \frac{v}{t}.$$

$$\therefore\ F = \frac{mv}{t}. \qquad \qquad (1)$$

The **momentum** of a body may be explained as the **quantity of motion,** and is measured by the product of the mass and velocity. Thus the momentum of the body in Fig. 64 is zero at A (where the velocity is zero) and is $mv$ at B. The momentum acquired in the interval $t$ seconds is $mv$; hence the momentum generated per second is $mv/t$. We may state therefore that **the applied force is equal numerically, to the rate of change of momentum, or to the momentum generated per second.**

The momentum generated, the acceleration, and the force applied have all the same direction and sense. These results are generalised in **Newton's second law of motion:**

**Rate of change of momentum is proportional to the applied force, and takes place in the direction in which the force acts.**

The dimensions of momentum are $ml/t$ or $mlt^{-1}$.

**Newton's third law of motion.**—**To every action there is always an equal and contrary reaction; or, the mutual actions of any two bodies are always equal and are oppositely directed.**

This law is the result of experience, of which a few instances may be noted. The hand of a person sustaining a load is subjected to a downward force—the weight of the load—and the hand applies an equal upward force. Similarly, a person applying a pull to a rope experiences an equal and opposite pull which the rope exerts on his hands. Equal and opposite forces applied in the same straight line to a body balance one another; under such conditions the body, if at rest, remains at rest, or, if in motion, will experience no change of motion.

A force applied to a body by means of some external agency, such as a pull along a string attached to the body, or a push from a rod in contact with the body, produces acceleration in accordance with the law $F = ma$. In this case the body, by virtue of its inertia, supplies a

reaction equal and opposite to the force applied to it by the external agency. In Fig. 65, F is the external force applied to the body · each particle of the body contributes to the equal opposite reaction by virtue of its inertia, and the total or resultant reaction is represented by the product $ma$. In fact, the equation $F=ma$ should be understood to represent the equality of two opposing forces, one, F, being the resultant external force applied to the body, and the other, $ma$, being an internal force produced by virtue of the inertia of the body.

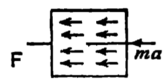


FIG. 65.—Resistance due to inertia.

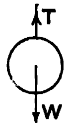


FIG. 66.

Should two external opposing forces T and W (Fig. 66) be applied to a body, unequal but in the same straight line, it is clear that a single external force $(W-T)$ would produce the same effect in changing the motion. $(W-T)$ may be called the **resultant external force**, and should be used as the value of F in the equation $F=ma$.

EXAMPLE 1.—What pull must be applied by a locomotive to give a train of 150 tons mass an acceleration of 1·5 feet per second per second if frictional resistances be neglected?

$$F=ma$$
$$=150\times 2240\times 1\cdot5=\underline{504,000}\text{ poundals}$$
$$=\frac{504000}{32\cdot2}=\underline{15,650}\text{ lb. weight.}$$

EXAMPLE 2.—Answer the same question if there are frictional resistances opposing the motion and amounting to 10 lb. weight per ton of train.

Total frictional resistance $=Q=150\times10=1500$ lb. weight.

Let the pull of the locomotive be P lb. weight, then the resultant force producing the acceleration will be $(P-Q)$ lb. weight. Hence

$$F=P-Q=\frac{ma}{g}=150\times2240\times1\cdot5\div32\cdot2$$
$$=15,650;$$
$$\therefore\ P=15,650+1500$$
$$=\underline{17,150}\text{ lb. weight.}$$

EXAMPLE 3.—Two bodies A and B (Fig. 67) are attached to the ends of a light cord passed over a pulley C. The cord may be assumed to be so fine that its mass may be neglected, and so flexible that the forces required in order to bend it round the pulley may be disregarded. It is assumed also that the pulley is so light that its mass may be neglected, and that its bearings are free from frictional resistance. Under these assumptions, the pulls in all parts of the cord will be equal, the pulley serving merely to change the direction of the cord. Take the masses of A and B to be $m_1$ and $m_2$ respectively, and discuss the motion.

Consider A; two external forces are applied to it, viz. the weight $m_1g$ and the upward pull T exerted by the cord. If these forces are equal, no motion will occur, or if there be motion, the velocity will be uniform. Suppose T to be larger than $m_1g$, then an upward acceleration $a$ will occur, and we may write:

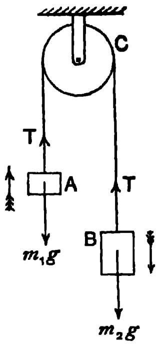


FIG. 67.—Motion under the action of gravity.

$$T - m_1g = m_1a. \quad \text{.......................(1)}$$

Now consider B; this body is subjected to a downward force $m_2g$ and an upward force T, and has a downward acceleration also equal to $a$ from the arrangement of the apparatus. Hence $m_2g$ is greater than T, and we may write:

$$m_2g - T = m_2a. \quad \text{.......................(2)}$$

Solving (1) and (2) in order to determine $a$ and T, we have, by addition:

$$m_2g - m_1g = (m_1 + m_2)a,$$

or,

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) g. \quad \text{...............(3)}$$

Dividing (1) by (2) gives:

$$\frac{T - m_1g}{m_2g - T} = \frac{m_1}{m_2};$$

$$\therefore \; m_2T - m_1m_2g = m_1m_2g - m_1T,$$

$$T(m_1 + m_2) = 2m_1m_2g,$$

$$T = \left(\frac{2m_1m_2}{m_1 + m_2}\right) g. \quad \text{.......................(4)}$$

It will be evident that, if $m_1$ and $m_2$ are equal, both bodies will have either no motion, or constant velocity; and the pull in the cord will be equal to the weight of one of the bodies.

This problem may be examined from another point of view. There are two bodies A and B, having a total mass $(m_1 + m_2)$ (Fig. 67); a resultant force acts, equal to the difference in their weights, viz. $(m_2g - m_1g)$; hence:

$$m_2g - m_1g = (m_1 + m_2)a,$$

or,

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) g,$$

which is the same result as that given in (3) above.

**Attwood's machine.**—In this machine an attempt is made to realise the conditions mentioned in Example 3 above by using a very light silk cord and a light aluminium pulley mounted on ball bearings, or bearings designed to eliminate friction so far as is possible. The machine is used in the following manner:

EXPT. 12.—**Use of Attwood's machine.** Equal loads A and B are hung from the ends of the cord (Fig. 68). A small additional load B′ is added and is adjusted so as to be just sufficient to overcome friction and to cause

B to have uniform downward velocity when given a slight start; A, of course, will have uniform upward velocity. Any additional weight D placed on B will produce acceleration in the whole of the moving parts. Denoting the masses of the loads by suffixes, we have, neglecting the masses of the pulleys and cord:

$$\text{Total mass in motion} = m_A + m_B + m_B' + m_D$$
$$= 2m_A + m_B' + m_D.$$

Force producing acceleration $= m_D g$;

$$\therefore \quad m_D g = (2m_A + m_B' + m_D)a,$$

or,
$$a = \frac{m_D g}{2m_A + m_B' + m_D}. \quad \text{...............(1)}$$

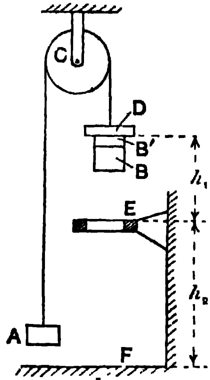


Fig. 68.—Attwood's machine.

To check this result we may employ the following method: A fixed ring is arranged at E, and has an internal diameter sufficiently large to permit of B and B′ passing through the ring, but will not so permit D. On arrival at E, D is arrested and the remaining moving parts will thereafter proceed with uniform velocity until they are brought to rest by B arriving at the fixed stop F. Measure $h_1$ and $h_2$; allow the motion to start unaided by any push, or otherwise, and start a stop watch simultaneously (a split-second stop watch is useful). Note the time at which D is arrested and also that at which B reaches F. Repeat several times, and take the mean time intervals. Let the mean time interval from the start to the instant at which D was arrested be $t_1$ and let the mean time interval in which B travels from E to F be $t_2$; also let the uniform velocity of B between E and F be $v$, and let the acceleration from the start until D is arrested be $a$.

Then
$$h_2 = vt_2, \quad \text{or} \quad v = \frac{h_2}{t_2}.$$

Also,
$$v = at_1, \quad \text{(p. 25)};$$

$$\therefore \quad a = \frac{v}{t_1} = \frac{h_2}{t_1 t_2}. \quad \text{..........................................(2)}$$

Or we may say:
$$h_1 = \tfrac{1}{2}at_1^2;$$

$$\therefore \quad a = \frac{2h_1}{t_1^2}. \quad \text{..............................................(3)}$$

Either of these expressions (2) or (3) may be used for the calculation of the acceleration, and the results should show fair agreement with that calculated from (1). It will be noted that this apparatus provides an experimental illustration of the truth of the law $F = ma$.

**Impulsive forces.**—Considering again the equation

$$F = \frac{mv}{t}, \quad \text{(p. 60)}, \quad \text{...............................(1)}$$

it will be noted that the principle involved is not affected by the magnitude of the interval of time. If this interval be very small, the conception of an **impulse** is obtained, *i.e.* a force acting during a very short time. Generally it is impossible to state the magnitude of such a force at any particular instant during the action, and the calculation of F from equation (1) gives the mean value of the force, and may be called the **average force of the blow.**

The equation may be written

$$Ft = mv. \qquad \qquad (2)$$

This form suggests plotting corresponding values of the force and time, should these be known, giving a diagram resembling that shown at OABC in Fig. 69. The average height of this diagram gives the average value of F. Owing to this method of deducing F from a diagram having a time base, the force F is sometimes called the **time average of the force.** This term is synonymous with the term average force of the blow.

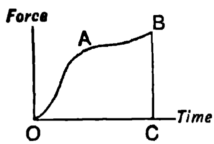

FIG. 69.—Time average of a force.

Since the average force F is represented by the mean height of the diagram in Fig. 69, and the base OC represents $t$, it follows that the area of the diagram represents F$t$. F$t$ may be called the **impulse of the force,** and is equal to the total change in momentum of the body.

EXAMPLE.—A bullet has a mass of 50 grams and a velocity of 400 metres per second. If it is brought to rest in 0·01 second, find the impulse and the average force of the blow.

$$\text{Impulse} = mv = 50 \times 400 \times 100$$
$$= 2 \times 10^6 \text{ gram cm./sec.}$$
$$\text{Average force of the blow} = \frac{mv}{t} = \frac{2 \times 10^6}{0{\cdot}01} = 2 \times 10^8 \text{ dynes}$$
$$= 20{\cdot}4 \times 10^4 \text{ gram weight.}$$

## Exercises on Chapter VI

**1.** Find the force required to give a mass of 15 pounds an acceleration of 45 feet per second per second.

**2.** A force of 9540 dynes acts on a mass of 2·5 kilograms. Find the acceleration.

**3.** Find factors for converting (*a*) dynes to poundals, (*b*) poundals to dynes.

**4.** A cycle and rider together have a mass of 190 pounds. When travelling at 10 miles per hour on a level road the cyclist ceases to pedal and observes that he comes to rest in a distance of 200 yards. Find the average resistance to motion.

**5.** A train has a mass of 200 tons. Starting from rest, a distance of 400 yards is covered in the first minute. Assuming that the acceleration was uniform, find the pull required to overcome the inertia of the train.

6. The cage of a lift has a mass of 1000 pounds. Find the pull in the rope to which the cage is attached (*a*) when the lift is descending at uniform speed, (*b*) when the lift is descending with an acceleration of 2 feet per second per second, (*c*) when the lift is ascending with the same acceleration.

7. A train has a mass of 250 tons. If the engine exerts a pull of 10 tons weight in producing an acceleration of 1 foot per second per second, find the resistance due to causes other than inertia.

8. A man who weighs 160 lb. slides down a rope, that hangs freely, with a uniform speed of 4 feet per second. What pull does he exert upon the rope, and what would happen if at a given instant he should reduce his pull by one half? L.U.

9. A mass of 10 pounds is placed upon a table and is connected by a thread which passes over a smooth peg at the edge with a mass of one pound that hangs freely. Assuming the table to be smooth, determine the velocity acquired by the two masses in one second, and find the tension in the thread. What would you infer if, in actual experiment, the masses were observed to move with uniform velocity; and what would be the tension in the thread in that case? L.U.

10. A fine cord passes over a pulley and has a mass of 0·5 kilogram hanging from one end and another mass of 0·9 kilogram hanging from the other end. Neglect friction and find the acceleration and the tension in the cord.

11. In an Attwood machine, a mass of 2 pounds is attached to each end of the cord. It is then found that an additional mass of 0·2 pound on one side is sufficient to maintain steady motion. Another mass of 0·4 pound is then placed on the same side and is found to produce a velocity of 4·72 feet per second at the end of a descent from rest of 4 feet. Is this result in accordance with the theory? Compare the actual acceleration with that given by the theory. $g = 32{\cdot}2$ feet per second per second.

12. A train moving with uniform acceleration passes three points A, B and C at 20, 30 and 45 miles an hour respectively. The distance AB is 2 miles. Find the distance BC. If steam is shut off at C and the brakes applied, find the total resistance in lb. weight per ton mass of the train in order that it may be brought to rest at a distance of one mile from C. L.U.

13. Find the momentum of a railway coach, mass 12 tons, travelling at 15 miles per hour. If the speed is changed to 12 miles per hour in 4 seconds, find the average resistance to the motion.

14. Find the impulse of a shot having a mass of 1200 pounds and travelling at 1500 feet per second. If the shot is brought to rest in 0·01 second, find the average force of the blow.

15. Define the terms 'acceleration', 'force', 'momentum', and state their precise relation to each other.

What is incorrect in the following expression:

(i) The force with which a body moves;

(ii) An acceleration of 10 feet per second? Adelaide University.

16. A mass of 2 pounds on a smooth table is connected by a string, passing over a light frictionless pulley at the edge of the table, with a suspended mass of 1 ounce. Find (*a*) the velocities of the masses after they have moved for 1 second from rest, and (*b*) the total momentum of the system at the same time. L.U.

**17.** Define the impulse of a force and an impulsive force. Find the direction and magnitude of a blow that will turn the direction of motion of a cricket ball weighing $5\frac{1}{2}$ oz., moving at 30 ft. per sec., through a right angle, and double its velocity. State in what units your answer is given. L.U.

**18.** A particle A of mass 10 oz. lies on a smooth table and is connected by a slack string which passes through a hole in the table with a particle B of mass 6 oz. lying on the ground directly beneath the hole in the table. A is projected along the table with a velocity of 8 feet per second. Find the impulsive tension when the string becomes taut and the common velocity of the particles immediately afterwards. Find also the height to which B will rise. L.U.

**19.** Explain what is meant by relative velocity. A ball of mass 8 ounces after falling vertically for 40 feet is caught by a man in a motor-car travelling horizontally at 30 miles an hour. Find the inclination to the vertical at which it will appear to him to be moving, and the magnitude of the impulse on the ball when it is caught. L.U.

**20.** Two masses of $\frac{1}{4}$ oz. and $7\frac{3}{4}$ oz., connected by an inextensible string 5 ft. long, lie on a smooth table $2\frac{1}{2}$ ft. high. The string being straight and perpendicular to the edge of the table, the lighter mass is drawn gently just over the edge and released. Find (*a*) the time that elapses before the first mass strikes the floor, and (*b*) the time that elapses before the second mass reaches the edge of the table. L.U.

**21.** State Newton's laws of motion, and show how from the first we obtain a definition of force, and from the second a measure of force.

A motor-car, running at the rate of 15 miles per hour, can be stopped by its brakes in 10 yards. Prove that the total resistance to the car's motion when the brakes are on is approximately one-quarter of the weight of the car. L.U.

**22.** A particle is projected up the steepest line of a smooth inclined plane, and is observed to pass downwards through a point 18 feet distant from the place of projection 4 seconds after passing upwards through the point. Further, there is an interval of 3 seconds between its transits through a point distant 32 feet from the place of projection. Find the velocity of projection and the slope of the plane. L.U.

**23.** Two masses of 5 lb. and 3 lb. at rest at a distance of 16 ft. apart are connected by a string in which a constant tension equal to the weight of 4 oz. is maintained. When will the masses meet, and how far will each have moved? C.W.B., H.C.

**24.** A long string fixed at one end to the ceiling passes down under a pulley of mass $m$ which it supports, then up over a fixed pulley also attached to the ceiling, and finally down to a weight of mass $m'$ which is hanging freely. Assuming that the hanging portions of the string are all vertical, prove that the upward acceleration of the mass $m'$ is $2(m-2m')g/(m+4m')$. C.W.B., H.C.

**25.** State Newton's three laws of motion. A particle hangs by a string from the roof of a railway carriage. Find the inclination of the string to the vertical when it is at relative rest in the carriage and the train is accelerating at the rate of 3 feet per second per second. J.M.B., H.S.C.

**26.** A smooth hemispherical bowl, of mass M, with centre C, lies rim downwards on a smooth table, and a particle, of mass $m$, is placed on it at a point A, whose angular distance from the vertex V of the bowl is $\alpha$. Show

that, if a horizontal force of suitable magnitude is applied to the bowl in the plane VCA, the particle will remain at rest relatively to the bowl as it moves. J.M.B., H.S.C.

**27.** A car whose mass is 2000 lb. starts from rest and the resistance to its motion is equal to 50 lb. wt. When it has travelled a distance $s$ feet the force exerted by the engine is F lb. wt. where

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $s$ | 0 | 10 | 20 | 30 | 40 | 50 | 60 |
| F | 644 | 634 | 622 | 607 | 587 | 565 | 537 |

Construct the acceleration-space graph of this (continuous) motion and find the speed of the car when it has travelled 60 feet. J.M.B., H.S.C.

# CHAPTER VII

## STATIC FORCES ACTING AT A POINT

**Specification of a force.**—In specifying a force the following particulars must be stated: (*a*) the point at which the force is applied; (*b*) the line of direction in which the force acts; (*c*) the sense along the line of direction; (*d*) the magnitude of the force.

Force is a vector quantity; this statement is confirmed by the fact that mass and acceleration are involved in the measurement of a force; mass is a scalar quantity, and acceleration is a vector quantity, hence force is also a vector quantity. It follows that two or more forces acting at a point and in the same plane may be compounded so as to give the **resultant force**, *i.e.* a force which has the same effect as the given forces, and the methods of vector addition explained in Chapter III may be employed.

It is convenient to speak of a 'force acting at a point', but this statement should not be taken literally. No material is so hard that it would not be penetrated if even a small force be applied to it at a mathematical point. What is meant is that the force may be imagined to be concentrated at the point in question without thereby affecting the condition of the body as a whole. Further, in speaking of a force applied at a point, it must not be forgotten that the mere existence of a force implies matter to which it is applied. In Fig. 70, a body A applies an action $P_A$ to another body B, and is itself subjected to a simultaneous and equal reaction $P_B$ applied by B.

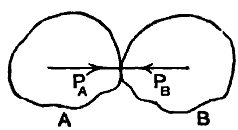

FIG. 70.—Action and reaction.

**Transmission of force along the line of action.**—In Fig. 71 a push P is applied to a body at a point A in a line BA. The general effect of P in producing changes of motion, or in maintaining the state of rest, will be unaltered if P be applied at any point O in the body and on the line of BA produced. There will, however, be alterations in the mutual actions between the particles of the body; it is clear that these will not be identical whether P is applied at A or O.

The mutual actions of the particles may be ignored in considering the state of rest or motion of the body as a whole. For example, a body A is subjected to pushes $P_B$, $P_C$ and $P_D$ applied by other three

bodies B, C and D at points *b*, *c* and *d* (Fig. 72). The three forces intersect at O and are in the plane of the paper. Disregarding the effects of the forces in producing actions between the particles of the

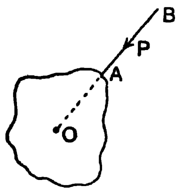


FIG. 71.—Transmission of a force.

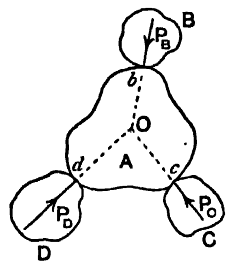


FIG. 72.—Three forces applied to a body.

body A, we may say that the effect on the body as a whole would be unaltered if the three forces were applied at O instead of at the given points *b*, *c* and *d*. In making this statement it is assumed that the body is **rigid**, *i.e.* its particles are assumed to adhere together so strongly as to prevent entirely any change in their relative positions. Otherwise relative motion of the parts of the body would occur independently of the motion of the body as a whole, and it is assumed that no such relative motion takes place.

**Stress.**—The term **stress** is given to the mutual actions which take place between one body and another, or between two parts of a body subjected to a system of forces. The term involves both of the dual aspects involved in force, and has therefore no sense. It thus becomes necessary to describe the action as **tensile stress** if the bodies, or the parts of the body, tend to separate; **compression stress** if they are forced together; and **shearing stress** if they tend to slide on one another. Stresses are discussed in more detail in Chapter XI.

**Parallelogram and triangle of forces.**—The **parallelogram of forces** is a construction similar to the parallelogram of velocities described on p. 34. Consider two forces, P and Q, acting at O and both in the plane of the paper (Fig. 73); to find the resultant, choose a suitable scale, and measure OA and OB to represent the magnitudes of P and Q respectively. Complete the parallelogram OACB, when the diagonal OC will represent the resultant R. In applying this construction, care must be taken that P and Q are arranged so that they act either both towards or both away from O. It will be remembered (p. 34) that the same arrangement must be made in the parallelogram of velocities.

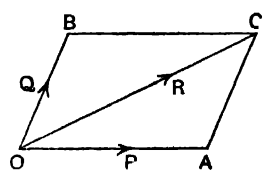


FIG. 73.—Parallelogram of forces.

The **triangle of forces** may be employed, and is similar to the triangle of velocities (p. 34). Given P and Q acting at O (Fig. 74); to find the resultant, draw AB to represent P, and BC to represent Q; then the resultant is represented by AC. Note that R does not act along AC (which may be anywhere on the paper), but at O, and is so shown in Fig. 74 by a line parallel to AC.

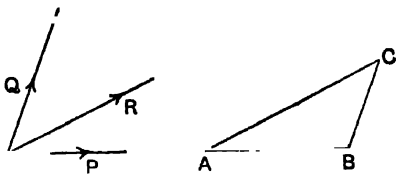


FIG. 74.—Triangle of forces.

**Forces acting in the same straight line.**—A body is said to be in **equilibrium** if the forces applied to it balance one another, *i.e.* produce no change in the state of rest or motion. Thus, if two equal and opposite forces P, P (Fig. 75), be applied at a point O in a body, both in the same straight line, they will balance one another, and the body is in equilibrium.


FIG. 75.—Two equal opposite forces.

If several forces in the same straight line act at a point in a body, the body will be in equilibrium if the sum of the forces of one sense is equal to the sum of those of opposite sense. Calling forces of one sense positive, and those of opposite sense negative, the condition may be expressed by stating that the algebraic sum of the given forces must be zero. Thus the forces $P_1$, $P_2$, $P_3$, etc. (Fig. 76), will balance, provided

$$P_1 + P_2 - P_3 - P_4 - P_5 = 0,$$

or

$$\Sigma P = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The symbol $\Sigma$ (sigma) means 'the algebraic sum of'; P placed after the symbol is taken to mean one only of a number of forces, of which P is given as a type. Equation (1) stated in words would read: The algebraic sum of all the forces of which P is a type is equal to zero.

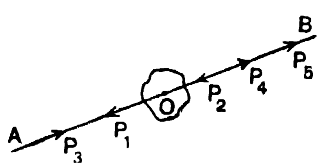


FIG. 76.—Forces in the same straight line.

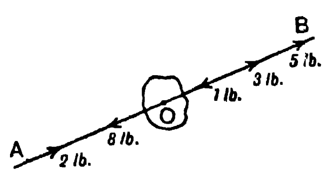


FIG. 77.

Should equation (1) give a numerical result which is not zero, it may be inferred that the given forces do not balance, but have a resultant with a magnitude equal to the calculated result. Equilibrium could be obtained by applying a force equal and opposite to

the resultant; this force is called the **equilibrant** of the system. Let R and E denote the resultant and equilibrant respectively, then

$$R = E.$$

The sense of R is positive or negative, depending upon whether the sum of the given positive forces is greater or less than that of the given negative forces. Thus, in Fig. 77, forces of sense from A towards B being called positive, we have

$$2 + 3 + 5 - 8 - 1 = +1.$$

Hence the given forces may be replaced by a single force of 1 lb. weight having the sense from A towards B. This result may be expressed by the equation

$$\Sigma P = R. \qquad \text{.......................................(2)}$$

**Three intersecting forces.**—Two forces whose lines of action intersect at a point may be balanced by first finding the resultant by means of the parallelogram, or triangle of forces; the resultant so found may be applied at the point instead of the given forces without altering the effect. The resultant so applied may then be balanced by applying an equilibrant equal and opposite to the resultant. Fig. 78 illustrates the method. Forces P and Q are given acting at O. In the triangle of forces *ab* represents P and *bc* represents Q; *ac* represents R, therefore *ca* represents the equilibrant E, which is now applied at O in a line parallel to *ca* and of sense given by the order of the letters *ca*

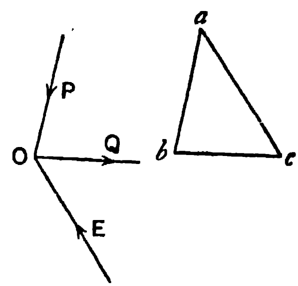


FIG. 78.—Triangle of forces applied to a push and a pull.

The conditions of balance of three intersecting forces may be formulated now: (*a*) **They must all act in one plane**; (*b*) **they must all act at one point**; (*c*) **they must be capable of representation by the sides of a closed triangle taken in order.**

The meaning of condition (*c*) may be understood by reference to the triangle of forces *abc* in Fig. 78. Here P, Q and E are represented respectively by *ab*, *bc* and *ca*; the order of these letters indicates the sense of each force; the figure is a closed triangle, and the perimeter has been traversed from *a* and back to *a* without it being necessary to reverse the direction in order to indicate the sense of any of the forces. Should the triangle of forces for three given forces fail to close, *i.e.* if a gap occurs between *a* and *a′* in Fig. 79, in which *ab*, *bc* and *ca′* represent the given forces, then we infer that the given forces are not in equilibrium.

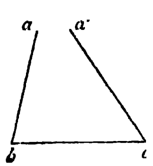


FIG. 79.

EXAMPLE.—Three given forces are known to be in equilibrium (Fig. 80); draw the triangle of forces.

This example is given to illustrate a convenient method of lettering the forces called **Bow's Notation.** The method consists in giving letters to the spaces instead of to the forces. In Fig. 80 (*a*), call the space between the 4 lb. and the 2 lb. A, that between the 2 lb. and the 3 lb. B, and the remaining space C. Starting in space A and passing into space B, a line AB (Fig. 80 (*b*)) is drawn parallel and proportional to the force crossed, and the letters are so placed that their order A to B represents the sense of that force. Now pass from space B into space C, and draw BC to represent completely the force crossed. Finish the construction by crossing from space C into space A, when CA in Fig. 80 (*b*) will represent the third force completely.

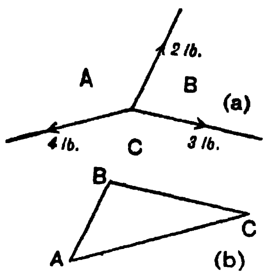


FIG. 80.—Application of Bow's Notation.

Examining these diagrams, it will be observed that a complete rotation round the point of application has been performed in Fig. 80 (*a*), and that there has been no reversal of the direction of rotation. Also that, in Fig. 80 (*b*), if the same order of rotation be followed, the sides represent correctly the senses of the various forces. Either sense of rotation may be used in proceeding round the point of application, clockwise or anti-clockwise, but once started there must be no reversal.

**Relation of forces and angles.**—In Fig. 81 (*a*) the three forces P, Q, S are in equilibrium, and ABC (Fig. 81 (*b*)) is the triangle of forces. We have

$$P : Q : S = AB : BC : CA.$$

Now $$AB : BC : CA = \sin\gamma : \sin\alpha : \sin\beta,$$

or $$P : Q : S = \sin\gamma : \sin\alpha : \sin\beta. \quad\text{.....................(1)}$$

The dotted lines in Fig. 81 (*a*) show that $\alpha$, $\beta$, $\gamma$ are respectively the

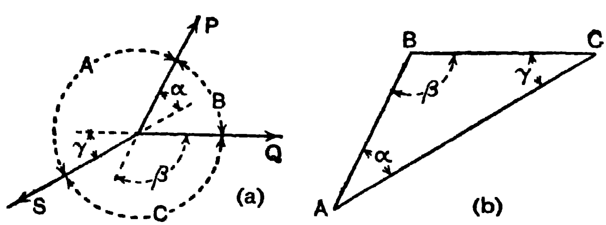


FIG. 81.—Relation of forces and angles.

angles between the produced directions of S and P, P and Q, Q and S; also the angles or spaces denoted by A, B, C in the same figure are the

supplements of these angles. Since the sine of an angle is equal to the sine of its supplement, we have, in Fig. 81 (*a*),

$$P : Q : S = \sin C : \sin A : \sin B. \quad \ldots\ldots(2)$$

Hence, if three intersecting forces are in equilibrium, **each force is proportional to the sine of the angle between the other two forces.**

**Rectangular components of a force.**—In the solution of problems it is often convenient to employ selected components of a force instead of the force itself. Generally these components are taken along two lines meeting at right angles on the line of the force; all three lines must be in the same plane. In Fig. 82, OC represents a given force P, and components are required along OA and OB which intersect at 90° at O. Complete the parallelogram of forces OBCA, which is a rectangle in this case, and let the angle COA be denoted by $\alpha$, then the components S and T are obtained as follows :

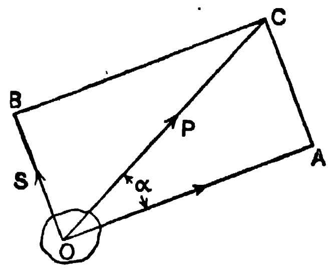


FIG. 82.—Rectangular components of a force.

$$AC = OB = OC \sin \alpha ;$$

$$\therefore \ S = P \sin \alpha. \quad \ldots\ldots(1)$$

$$OA = OC \cos \alpha ;$$

$$\therefore \ T = P \cos \alpha. \quad \ldots\ldots(2)$$

$$OC^2 = AC^2 + OA^2 = OB^2 + OA^2 ;$$

$$\therefore \ P^2 = S^2 + T^2. \quad \ldots\ldots(3)$$

**Relation of the resultant and inclined components.**—In Fig. 83 components P and Q are given, and the resultant R has been found by means of the parallelogram of forces OACB. From trigonometry, we have

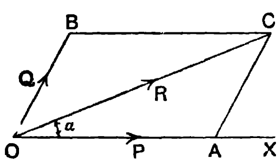


FIG. 83.—Inclined components of a force.

$$OC^2 = OA^2 + AC^2 - 2 \,.\, OA \,.\, AC \,.\, \cos OAC.$$

Also, $AC = OB$,

and $\cos OAC = -\cos CAX = -\cos AOB$ ;

$$\therefore \ OC^2 = OA^2 + OB^2 + 2 \,.\, OA \,.\, OB \,.\, \cos AOB,$$

or, $$R^2 = P^2 + Q^2 + 2PQ \cos AOB. \quad \ldots\ldots(1)$$

To find the angle $\alpha$ which R makes with OA, we have

$$\frac{Q}{R} = \frac{\sin \alpha}{\sin OAC} = \frac{\sin \alpha}{\sin CAX} = \frac{\sin \alpha}{\sin AOB} ;$$

$$\therefore \ \sin \alpha = \frac{Q}{R} \sin AOB. \quad \ldots\ldots(2)$$

EXAMPLE 1.—A particle of weight W is kept at rest on a smooth plane inclined at an angle $\alpha$ to the horizontal, (*a*) by a force parallel to the plane, (*b*) by a horizontal force. Find each of these forces.

The term **smooth** is used to indicate a surface incapable of exerting any frictional forces. Such a surface, if it could be realised, would be unable to exert any action on a body in contact with it in any line other than the normal to the surface at the point of contact.

(*a*) In Fig. 84 (*a*), W is represented by *ab*; the required force P, and the normal reaction of the plane R, are represented respectively in the triangle

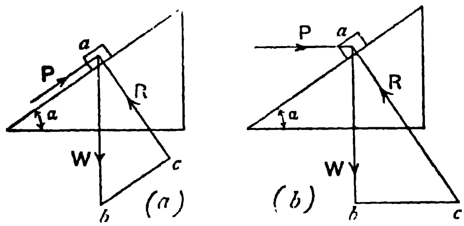


FIG. 84.—Particles on smooth inclines.

of forces *abc* by *bc* and *ca*. Since these lines are respectively parallel and at right angles to the plane, *bca* is a right angle; also the angle $bac = \alpha$.

$$\therefore \frac{P}{W} = \frac{bc}{ab} = \sin \alpha ;$$

$$\therefore P = W \sin \alpha.$$

R may be found thus:

$$\frac{R}{W} = \frac{ca}{ab} = \cos \alpha ;$$

$$\therefore R = W \cos \alpha.$$

(*b*) In this case the triangle of forces is the right-angled triangle *abc* (Fig. 84 (*b*)).

$$\frac{P}{W} = \frac{bc}{ab} = \tan \alpha ;$$

$$\therefore P = W \tan \alpha.$$

Also,

$$\frac{R}{W} = \frac{ca}{ab} = \sec \alpha ;$$

$$\therefore R = W \sec \alpha.$$

EXAMPLE 2.—A particle of weight W is kept at rest on a smooth plane inclined at an angle $\alpha$ to the horizontal by means of a force P inclined at an angle $\beta$ to the plane (Fig. 85). Find P and the reaction of the plane.

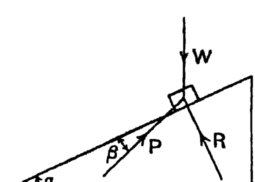


FIG. 85.

In Fig. 85, the angle between P and R is $(90° - \beta)$; also the angle between W and R is $(180° - \alpha)$. Hence (p. 73)

$$\frac{P}{W} = \frac{\sin (180 - \alpha)}{\sin (90 - \beta)} = \frac{\sin \alpha}{\cos \beta} ;$$

$$\therefore P = W \sin \alpha / \cos \beta.$$

The angle between P and W is $(90° + \alpha + \beta)$.

$$\therefore \frac{R}{W} = \frac{\sin(90 + \alpha + \beta)}{\sin(90 - \beta)} = \frac{\cos(\alpha + \beta)}{\cos\beta}$$

$$= \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\beta};$$

$$\therefore R = W(\cos\alpha - \sin\alpha\tan\beta).$$

**Systems of uniplanar concurrent forces.**—In Fig. 86, $P_1$, $P_2$, $P_3$, $P_4$ are four typical forces, all in the same plane and intersecting at the same

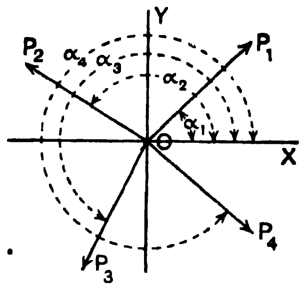


FIG. 86.—System of uniplanar forces acting at a point.

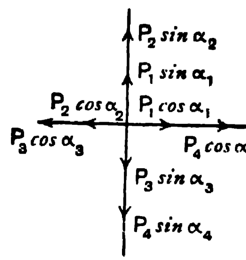


FIG. 87.—First step in the reduction of the system.

point O. OX and OY are two axes in the same plane as the forces and intersect at O at 90°. The angles of direction of the forces are stated with reference to OX, and are denoted by $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$. In order to take advantage of the usual conventions regarding the algebraic signs of sines and cosines, the given forces should be arranged so as to be either all pulls or all pushes. Taking components along OX and OY (Fig. 87), we have :

Components along OX ; $P_1\cos\alpha_1$, $P_2\cos\alpha_2$, $P_3\cos\alpha_3$, $P_4\cos\alpha_4$.

Components along OY ; $P_1\sin\alpha_1$, $P_2\sin\alpha_2$, $P_3\sin\alpha_3$, $P_4\sin\alpha_4$.

Taking account of the algebraic signs, the components along OX towards the right are positive and the others are negative. Similarly, the components acting upwards along OY are positive and the others are negative. The resultants $R_X$ and $R_Y$ of the components in OX and OY respectively are given by :

$$P_1\cos\alpha_1 + P_2\cos\alpha_2 + P_3\cos\alpha_3 + P_4\cos\alpha_4 = R_X,$$

$$P_1\sin\alpha_1 + P_2\sin\alpha_2 + P_3\sin\alpha_3 + P_4\sin\alpha_4 = R_Y.$$

Or, using the abbreviated method of writing these,

$$\Sigma P\cos\alpha = R_X, \quad \text{.................................(1)}$$

$$\Sigma P\sin\alpha = R_Y. \quad \text{.................................(2)}$$

The given system has thus been reduced to two forces $R_X$, $R_Y$, as shown in Fig. 88. To find the resultant R, we have

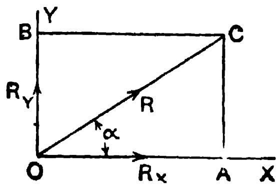


FIG. 88. Resultant of the system shown in Fig. 86.

$$R = \sqrt{R_X{}^2 + R_Y{}^2}, \quad \text{.............(3)}$$

$$\tan \alpha = \frac{CA}{OA} = \frac{OB}{OA} = \frac{R_Y}{R_X}. \quad \text{...........(4)}$$

The given system of forces will be in **equilibrium** if both $R_X$ and $R_Y$ are zero. The algebraic conditions of equilibrium are obtained from (1) and (2) :

$$\Sigma P \cos \alpha = 0, \quad \text{...............(5)}$$

$$\Sigma P \sin \alpha = 0. \quad \text{...............(6)}$$

This pair of simultaneous equations may be used for the solution of any problem regarding the equilibrium of any system of uniplanar concurrent forces.

**Graphical solution by the polygon of forces.**—By application of the principle of vector addition, the equilibrium of a number of uniplanar forces acting at a point may be tested. Four such forces are given in Fig. 89 (*a*), and are described by Bow's notation (p. 72). Starting

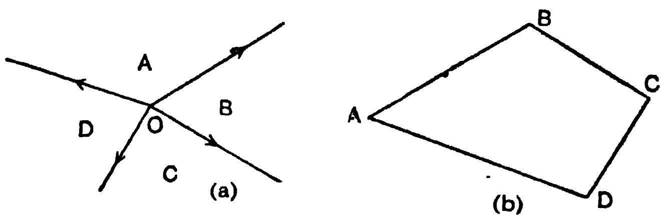


FIG. 89.—Polygon of forces.

in space A and going round O clockwise, lines are drawn in Fig. 89 (*b*) representing completely each force crossed. The production of a closed polygon ABCD is sufficient evidence that the forces are in equilibrium ; a gap would indicate that the forces have a resultant, which would be represented by the line required to close the gap, and the equilibrant of the system would be equal and opposite to the resultant. Fig. 89 (*b*) is called the **polygon of forces.**

We may therefore state that a system of uniplanar forces acting at a point will be in equilibrium, provided **a closed polygon can be drawn in which the sides taken in order represent completely the given forces.**

**Concurrent forces not in the same plane.**—Most of the cases of forces not in the same plane are beyond the scope of this book. The following exercise indicates the manner in which simple cases of such forces may be solved.

EXAMPLE.—In Fig. 90 is shown the plan and elevation of a wedge. The top surface is smooth and makes an angle of 30° with the horizontal. A particle A of weight W is kept at rest on the wedge by means of two light cords AB and AC, fastened at B and C, and making angles of 45° and 30° respectively with the line of greatest slope DE. Find the pulls $T_1$ and $T_2$ in AB and AC respectively.

(*N.B.*—The actual sizes of the angles of 45° and 30° cannot be seen in the plan in Fig. 90.)

Resolve $T_1$ and $T_2$ into components along DAE and along a horizontal axis AZ at 90° to DAE. The components along DAE are $T_1 \cos 45°$ and $T_2 \cos 30°$, both of the same sense: those along AZ are $T_1 \sin 45°$ and $T_2 \sin 30°$, and are of opposite sense (Fig. 90, plan).

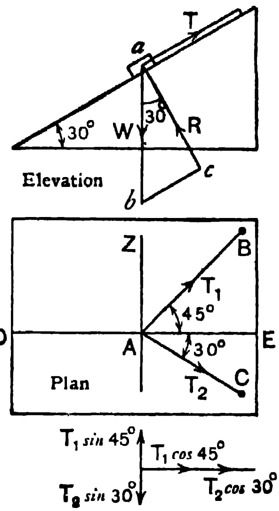


FIG. 90.

For equilibrium in the direction of AZ we have:

$$T_1 \sin 45° = T_2 \sin 30°,$$

or,
$$\frac{T_1}{\sqrt{2}} = \tfrac{1}{2} T_2 \text{ ..................... (1)}$$

Let $T = T_1 \cos 45° + T_2 \cos 30° = \frac{T_1}{\sqrt{2}} + \frac{T_2\sqrt{3}}{2}$ (Fig. 90, elevation).

Then T, W and the reaction of the plane R are in equilibrium, and *abc* is the triangle of forces (Fig. 90, elevation). Hence

$$\frac{T}{W} = \frac{bc}{ab} = \sin 30° = \tfrac{1}{2},$$

$$\therefore\ T = \tfrac{1}{2} W,$$

or,
$$\frac{T_1}{\sqrt{2}} + \frac{T_2\sqrt{3}}{2} = \tfrac{1}{2} W. \text{ ................................................ (2)}$$

Substituting from (1), we have

$$\tfrac{1}{2} T_2 + \frac{T_2\sqrt{3}}{2} = \tfrac{1}{2} W, \quad \therefore\ T_2 = \frac{W}{1 + \sqrt{3}}. \text{ ........................ ...(3)}$$

And from (1),
$$T_1 = \frac{\sqrt{2}}{2} T_2 = \frac{W\sqrt{2}}{2(1 + \sqrt{3})}. \text{ ................................(4)}$$

EXPT. 13.—**Parallelogram of forces.** In Fig. 91 is shown a board attached to a wall and having three pulleys A, B and C capable of being clamped to any part of the edge of the board. These pulleys should run easily. Pin a sheet of drawing paper to the board. Clamp the pulleys A and B in any given positions. Tie two silk cords to a small ring, pass a bradawl through the ring into the board at O, and lead the cords over the pulleys at A and B. The ends of the cords should have scale pans attached, in

which weights may be placed. Thus, known forces P and Q are applied to the ring at O. Take care in noting these forces that the weight of the scale pan is added to the weight you have placed in it. Mark carefully the directions of P and Q on the paper, and find their resultant R by means of the parallelogram O*abc*. Produce the line of R, and by means of a third cord tied to the ring, apply a force E equal to R, bringing the cord exactly into the line of R by using the pulley C clamped to the proper position on the board. Note that the proper weight to place in the scale pan is E less the weight of the scale pan, so that weight and scale pan together equal E. If the method of construction is correct, the bradawl may be withdrawn without the ring altering its position.

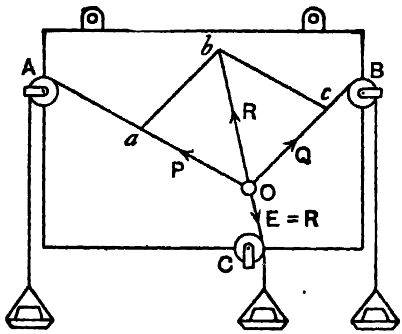

FIG. 91.—Apparatus for demonstrating the parallelogram of forces.

In general it will be found that, after the bradawl is removed, the ring may be made to take up positions some little distance from O. This is due to the friction of the pulleys and to the stiffness of the cords bending round the pulleys, giving forces which cannot easily be taken into account in the above construction.

EXPT. 14.—**Pendulum.** Fig. 92 (*a*) shows a pendulum consisting of a heavy bob at A suspended by a cord attached at B and having a spring

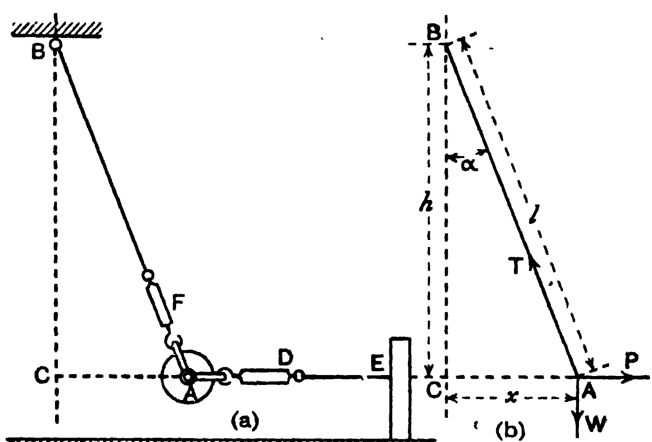

FIG. 92.—Experiment on a pendulum.

balance at F. Another cord is attached to A and is led horizontally to E, where it is fastened; a spring balance at D enables the pull to be read.

Find the pulls T and P of the spring balances F and D respectively when A is at gradually increased distances $x$ from the vertical BC. Check these by calculation as shown below, and plot P and $x$.

Since P, W and T are respectively horizontal, vertical and along AB, it follows that ABC is the triangle of forces for them. Hence

$$\frac{\mathsf{P}}{\mathsf{W}}=\frac{\mathsf{CA}}{\mathsf{BC}}=\frac{x}{h} \text{ (Fig. 92 (b)),}$$

$$\mathsf{P}=\frac{x}{h}\mathsf{W}=\mathsf{W}\tan\alpha. \qquad (1)$$

Also,

$$\frac{\mathsf{T}}{\mathsf{W}}=\frac{\mathsf{AB}}{\mathsf{BC}}=\frac{l}{h},$$

$$\mathsf{T}=\frac{l}{h}\mathsf{W}=\mathsf{W}\sec\alpha. \qquad (2)$$

Measure $l$, also $x$ and $h$, for each position of the bob, and calculate P and T by inserting the required quantities in (1) and (2). Tabulate thus :

Weight of bob in kilograms = W =

Length of AB in cm. = $l$ =

| $x$ cm. | $h$ cm. | Calculated values, in kilograms | | Observed values from spring balances, kilograms | |
|---|---|---|---|---|---|
| | | $\mathsf{P}=\frac{x}{h}\mathsf{W}$ | $\mathsf{T}=\frac{l}{h}\mathsf{W}$ | P | T |
| | | | | | |

The curve will resemble that shown in Fig. 93. Note how nearly straight it is for comparatively small values of $x$.

EXPT. 15.—**Polygon of forces.** In Fig. 94 is shown the finished results of an experiment on the polygon of forces. The apparatus and method

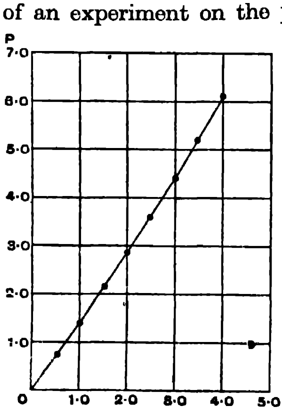


FIG. 93.—Graph of P and $x$ for a pendulum.

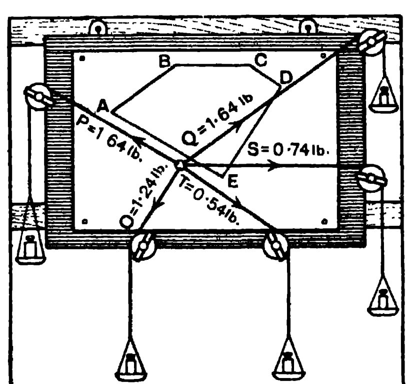


FIG. 94.—An experiment on the polygon of forces.

are similar to that employed in Expt. 13 (p. 77). Four forces have been assumed, and the equilibrant has been found from the closing side of the polygon ABCDEA. On application of the equilibrant, the bradawl may be withdrawn without the ring moving.

EXPT. 16.—**Derrick crane.** A derrick crane model is shown in Fig. 95, consisting of a post AB firmly fixed to a base board which is screwed to a table; a jib AC has a pointed end at A bearing in a cup recess, a pulley at C and a compression spring balance at D. A tie BC supports the jib and is of adjustable length; a spring balance for measuring the pull is inserted at F. The weight is supported by a cord led over the pulley at C and attached to one of the screw-eyes on the post. The inclination of the jib may be altered by adjusting the length of BC, and the inclinations of EC and BC may be changed by making use of different screw-eyes.

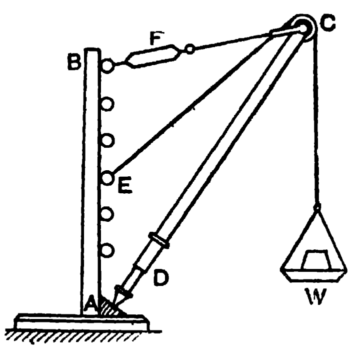


FIG. 95.—Model derrick crane.

Observe the spring balances and so find the push in the jib and the pull in the tie for different values of W and different dimensions of the apparatus. Check the results by means of the polygon of forces, constructed as follows:

First measure the dimensions AB, BC, AC and AE, and construct an outline diagram of the crane (Fig. 96 (*a*)). It may be assumed that the pulley at

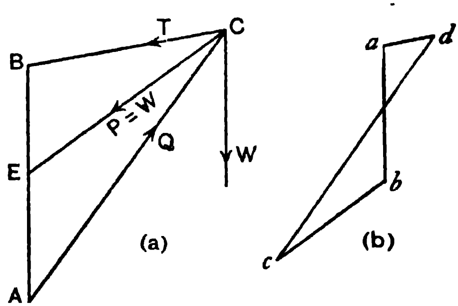


FIG. 96.—Forces in a derrick crane.

C merely changes the direction of the cord without altering the force in it; hence P = W (Fig. 96 (*a*)). The polygon of forces (Fig. 96 (*b*)) is drawn by making *ab* represent W, and *bc* represent P; lines are then drawn from *a* parallel to T, and from *c* parallel to Q; these intersect at *d*. Q and T may be scaled from *cd* and *da* respectively.

The values of Q and T so found will not agree very well with those shown by the spring balances. This is owing to the weights of the parts of the apparatus not having been taken into account. Approximate corrections may be applied to the spring balance readings by removing W from the scale pan and noting the readings of the spring balances; these will give the forces in the jib and tie produced by the weights of the parts, and should be deducted from the former readings, when fair agreement will be found with the results obtained from the polygon of forces.

## Exercises on Chapter VII

**1.** A nail is driven into a board and two strings are attached to it. If the angle between the strings is 60°, both strings being parallel to the board, and if one string is pulled with a force equal to 4 lb. weight and the other with a force of 8 lb. weight, find by construction the resultant force on the nail.

**2.** Answer Question 1 supposing a rod is substituted for the first string and is pushed with a force equal to 4 lb. weight.

**3.** One component of a force of 3 kilograms weight is equal to 2 kilograms weight, and the angle between this component and the force is 40°. Find the other component by construction.

**4.** The components of a force of 10 lb. weight are 5 lb. weight and 7 lb. weight respectively. Find by construction the lines of action of the components.

**5.** A pull of 6 lb. weight and another force Q of unknown magnitude act at a point, their lines of action being at 90°; they are balanced by a force of 8 lb. weight. Calculate the magnitude of Q and the angle between Q and the force of 8 lb. weight.

**6.** Answer Question 5 if Q and the force of 6 lb. weight intersect at 60°.

**7.** Two pulls of 10 lb. weight each act at a point. Find the equilibrant by calculation in the cases when the angle between the pulls is 165°, 170°, 174°, 178°, 180°. Plot a curve showing the relation of the magnitude of the equilibrant and the angle between the pulls.

**8.** A particle weighing 2 lb. is kept at rest on a smooth plane inclined at 40° to the horizontal by a force P. Calculate the magnitude of P when it is (*a*) parallel to the plane, (*b*) horizontal, (*c*) pulling at 20° to the plane, (*d*) pushing at 30° to the plane. In each case find the reaction R of the plane.

**9.** A particle of mass $m$ pounds slides down a smooth plane inclined at 25° to the horizontal. Find the resultant force producing acceleration; hence find the acceleration and the time taken to travel a distance of 8 feet from rest.

**10.** Two strings of lengths $3\frac{1}{4}$ feet and $3\frac{3}{4}$ feet are tied to a point of a body whose weight is 8 lb., and their free ends are then tied to two points in the same horizontal line $3\frac{1}{2}$ feet apart. Find the tension in each string. L.U.

**11.** A kite having a mass of 2 pounds is flying at a vertical height of 100 feet at the end of a string 220 feet long. If the tension of the string is equal to a weight of $1\frac{1}{2}$ lb., find graphically the magnitude and direction of the force of the wind on the kite.

**12.** Three forces P, Q, E are in equilibrium. P = Q, and E = 1·25 P. Find the angle between the directions of P and Q. Answer the same question if P = Q = E.

**13.** A rope is fastened to two points A, B, and carries a weight of 50 lb. which can slide smoothly along the rope. The coordinates of B with respect to horizontal and vertical axes at A are 8 feet and 1·2 feet, and the length of the rope is 10 feet. Find graphically the position of equilibrium and the tension in the rope. L.U.

**14.** In Fig. 97 is shown a bent lever ABC, pivoted at C. The arms CA and CB are at 90° and are 15 inches and 6 inches respectively. A force P of 35 lb. weight is applied at A at 15° to the horizontal, and another Q is applied at B at 20° to the vertical. Find the magnitude of Q and the magnitude and direction of the reaction at C required to balance P and Q. Neglect the weight of the lever.

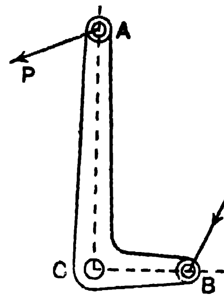


FIG. 97.

**15.** Two scaffold poles AB and AC stand on level ground in a vertical plane, their tops being lashed together at A. AB is 20 feet, AC is 15 feet and BC is 15 feet. Find the push in each pole when a load of 1 ton weight is hung from A.

**16.** The jib of a derrick crane measures 19 feet, the tie is 17½ feet and the post is 9 feet long. A load of 2·5 tons weight is attached to a chain which passes over a single pulley at the top of the jib and then along the tie. Find the push in the jib and the pull in the tie. Neglect friction and the weights of the parts of the crane.

**17.** Answer Question 16 supposing the chain, after leaving the pulley at the top of the jib, passes along the jib.

**18.** Four loaded bars meet at a joint as shown in Fig. 98. P and Q are in the same horizontal line; T and W are in the same vertical; S makes 45° with P. If P = 15 tons weight, W = 12 tons weight, S = 6 tons weight, find Q and T.

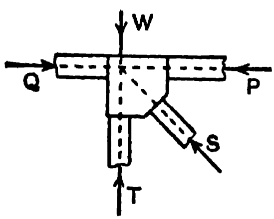


FIG. 98.

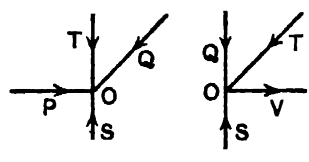


FIG. 99.

**19.** Lines are drawn from the centre O of a hexagon to each of the corners A, B, C, D, E, F. Forces are applied in these lines as follows: From O to A, 6 lb. weight; from B to O, 2 lb. weight; from C to O, 8 lb. weight; from O to D, 12 lb. weight; from E to O, 7 lb. weight; from F to O, 3 lb. weight. Find the resultant.

**20.** In Fig. 99 forces in equilibrium act at O as follows: In the front elevation, P, Q and S are in the plane of the paper and T is at 45° to the plane of the paper; Q makes 135° with S. In the side elevation, T and V are in the plane of the paper; V is perpendicular to the plane containing

P, Q and S, and T makes 45° with V. Given Q = 40 tons weight, T = 25 tons weight, find P, S and V.

**21.** State and establish the proposition known as the polygon of forces.

OA, OB, OC, OD, OE are five bars in one plane meeting at O, the angles AOB, BOC, COD, DOE being each 30°. Forces of 1, 2, 3, 4 and 5 tons respectively act outwards from O along the bars. The joint O is held in equilibrium by two other bars pulling in the opposite directions to OA and OD. Find the pull along each of these bars.

**22.** A particle of weight W is kept in equilibrium on a smooth inclined plane (angle of inclination = $\theta$) by a single force parallel to the plane. Find the magnitude of the force. If the particle is kept in equilibrium by three forces P, Q, R, each parallel to the plane and inclined at angles $\alpha$, $\beta$, $\gamma$ to the line of greatest slope, find all the relations existing between P, Q, R, W, $\alpha$, $\beta$, $\gamma$, $\theta$.

**23.** Enunciate the polygon of forces and show how it may be used to find the resultant of a number of concurrent forces. Explain also the method of getting the resultant by considering the resolved parts of the forces in two directions at right angles.

**24.** A string 12 feet long has 11 knots at intervals of 1 foot. The ends of the string are tied to two supports, A, B, 9 feet apart and in the same horizontal line. A load of 4 lb. weight is suspended from each knot in turn; find the tensions in the part of the string attached to A. Plot these tensions as ordinates, and horizontal distances of the load from A as abscissae.

**25.** A building measures 40 feet long and 20 feet wide. In the plan, the ridge of the roof is parallel to the long sides and bisects the short sides. The ridge is 5 feet higher than the eaves. Wind exerts a normal pressure of 40 lb. weight per square foot on one side of the roof. Find the horizontal and vertical components of the total force exerted by the wind on this side.

**26.** A string of length 2 feet is attached to two points A and B at the same level and at a distance 1 foot apart. A ring of 10 lb. wt., slung on to the string, is acted on by a horizontal force P which holds it in equilibrium vertically below B. Find the tension in the string and the magnitude of P. J.M.B., H.S.C.

**27.** Determine the position of a point D in a plane through three given points A, B, C, such that forces represented by $m$ . AB, $n$ . AC and $(m+n)$DA acting at the point A along AB, AC and DA respectively are in equilibrium.

O is any point on a median of a triangle ABC. Forces act from O towards A, B and C proportional to the distances of O from these points. Prove that their resultant is represented in magnitude and direction by 3 . OG, where G is the centroid of the triangle. J.M.B., H.S.C.

**28.** A heavy spherical ball of weight W lb. rests in a V-shaped trough, whose sides are inclined at angles of $\alpha$ and $\beta$ to the horizontal. Find the pressure on each side. If a second equal ball be placed on the side of inclination $\alpha$ so as to rest above the first, find the pressure of the lower ball on the two sides. J.M.B., H.S.C.

**29.** A body of mass 10 pounds is on a smooth plane inclined at 30° to the horizontal. A light string attached to it lies along a line of greatest slope, passes over a smooth light pulley at the top of the plane and has a mass of 3 pounds hanging from the other end. Find the acceleration of the system and the stress on the axle of the pulley. J.M.B., H.S.C.

## CHAPTER VIII

### MOMENTS. PARALLEL FORCES

**Moment of a force.**—The **moment of a force** is the tendency of the force to rotate the body to which it is applied, and is measured by the product of the magnitude of the force and the length of a line drawn from the axis of rotation perpendicular to the line of the force. Thus, in Fig. 100, is shown a body capable of rotating about an axis passing through O and perpendicular to the plane of the paper. A force P is applied in the plane of the paper and its moment is measured by

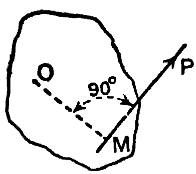


FIG. 100.—Moment of a force.

$$\text{Moment of } P = P \times OM,$$

OM being drawn from O at right angles to P.

Moments involve both the units of force and of length employed in the calculation. In the C.G.S. system moments may be measured in dyne-centimetres or gram-weight-centimetres ; in the British system, poundal-feet or lb.-weight-feet are customary. The dimensions of the moment of a force are obtained by taking the product of the dimensions of force and length, thus :

$$\text{Dimensions of the moment of a force} = \frac{ml}{t^2} \times l = \frac{ml^2}{t^2}, \quad \text{or} \quad ml^2t^{-2}.$$

The sense of the moment of a force is described as **clockwise** or **anti-clockwise**, according to the direction of rotation which would result from the action of the force. In calculations it is convenient to describe moments of one sense as positive ; those of contrary sense will then be negative.

It is evident that no rotation can result from the action of a force which passes through the axis of rotation ; such a force has no moment.

**Representation of a moment.**—In Fig. 101 is shown a body free to rotate about O and acted on by a force P, represented by the line AB. Draw OM perpendicular to AB, producing AB if necessary. Join OA and OB. Then

$$\text{Moment of } P = P \times OM = AB \times OM$$
$$= 2\triangle OAB.$$

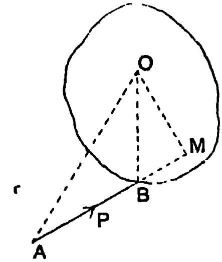


FIG. 101.—Representation of a moment.

We may therefore take twice the area of the triangle, formed by joining the extremities

of the line representing the force to the point of rotation, as a measure of the moment of the force.

**The components of a force have equal opposite moments about any point on the line of the resultant.**—In Fig. 102, R acts at O, and has components P and Q given by the parallelogram of forces OBDC. A is any point on the line of R, and AM and AN are perpendicular to P and Q respectively. $\alpha$ and $\beta$ are the angles between R and P, and R and Q.

$$\frac{\text{Moment of P}}{\text{Moment of Q}}=\frac{\text{P}\times\text{AM}}{\text{Q}\times\text{AN}}=\frac{\text{P}\times\text{OA}\sin\alpha}{\text{Q}\times\text{OA}\sin\beta}$$

$$=\frac{\text{P}\sin\alpha}{\text{Q}\sin\beta}.$$

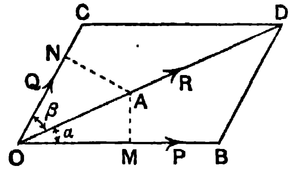


FIG. 102.—Moments of P and Q.

Also, $$\frac{\text{P}}{\text{Q}}=\frac{\text{OB}}{\text{OC}}=\frac{\text{OB}}{\text{BD}}=\frac{\sin\beta}{\sin\alpha};$$

$$\therefore\ \frac{\text{moment of P}}{\text{moment of Q}}=\frac{\sin\beta\sin\alpha}{\sin\alpha\sin\beta}=1.$$

$$\therefore\ \text{moment of P}=\text{moment of Q}.$$

**The moment of a force about any point is equal to the algebraic sum of the moments of its components.**—There are two cases, one in which the point is so chosen that the components have moments of the same sign (Fig. 103 (*a*)); in the other case (Fig. 103 (*b*)) the components have moments

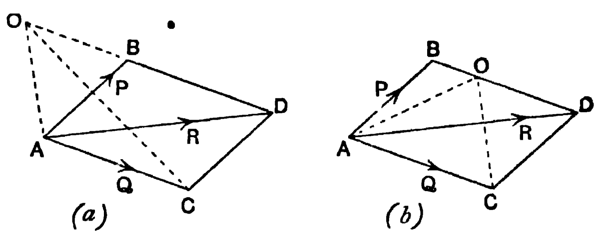


FIG. 103.—Moments of P, Q and R about O.

of opposite sign. In each figure let R be the given force and let O be the point of rotation. Let the components be P and Q, and draw OBD parallel to Q, and cutting P and R in B and D respectively. Complete the parallelogram ABDC, then

$$\text{P}:\text{Q}:\text{R}=\text{AB}:\text{AC}:\text{AD}$$

Join OA and OC. Then, in Fig. 103 (*a*),

$$\triangle\text{OAB}+\triangle\text{ABD}=\triangle\text{AOD}.$$

Also, $$\triangle\text{ABD}=\triangle\text{ACD}=\triangle\text{OAC}.$$

$$\therefore\ \triangle\text{OAB}+\triangle\text{OAC}=\triangle\text{OAD},$$

or, moment of P + moment of Q = moment of R (p. 84).

*In Fig. 103 (b)* we have

$$\Delta OAB + \Delta OAD = \Delta ABD.$$

Also, $$\Delta ABD = \Delta ACD = \Delta OAC;$$

$$\therefore \Delta OAB + \Delta OAD = \Delta OAC,$$

or, $$\Delta OAD = \Delta OAC - \Delta OAB,$$

or, moment of R = moment of Q − moment of P.

Hence, in taking moments, we may substitute either the components for the resultant, or the resultant for the components, without altering the effect on the body.

**Principle of moments.**—Let a number of forces, all in the same plane, act on a body free to rotate about a fixed axis. If no rotation occurs, then **the sum of the clockwise moments is equal to the sum of the anticlockwise moments.**

This **principle of moments** may be understood by taking any two of the forces, both having clockwise moments. The moment of the resultant of these forces is equal to the sum of the moments of the forces. Take this resultant together with another of the given forces having a clockwise moment; the moment of these will again be equal to the sum of the moments. Repeating this process gives finally a single force having a clockwise moment equal to the sum of all the given clockwise moments.

Treating the forces having anticlockwise moments in the same manner gives a single force having an anticlockwise moment equal to the sum of all the given anticlockwise moments. Hence the resultant of the two final forces has a moment equal to the algebraic sum of the given clockwise and anticlockwise moments, and if these be equal the resultant moment is zero and no rotation will occur.

EXPT. 17.—**Balance of two equal opposing moments.** In Fig. 104, a rod AB has a hole at A through which a bradawl has been pushed into a vertical

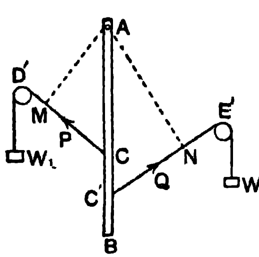


FIG. 104.—Two inclined forces, having equal opposing moments.

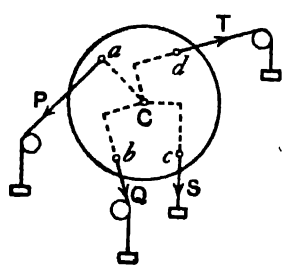


FIG. 105.—Disc in equilibrium under the action of several forces.

board. The rod AB hangs vertically and can turn freely about A. Fix it in this position by pushing another bradawl through a hole near B. Attach a fine cord at C, lead it over a pulley D′ and attach a weight $W_1$,

thus applying a pull $P = W_1$ at C. Measure the perpendicular AM drawn from A to P, and calculate the moment of $P = P \times AM$. Attach another fine cord at C′ and lead it over a pulley E′. Measure the perpendicular AN, drawn from A to the cord, and calculate Q from

$$\text{Moment of } Q = \text{Moment of } P,$$
$$Q \times AN = P \times AM,$$
$$Q = \frac{P \times AM}{AN}.$$

Apply a weight $W_2$ equal to the calculated value of Q, and withdraw the bradawl at B. If the rod remains vertical, the result may be taken as evidence of the principle that two equal opposing moments balance.

EXPT. 18.—**Principle of moments.** In Fig. 105 is shown a wooden disc which can turn freely about a bradawl pushed through a central hole into a vertical board. Apply forces at *a, b, c, d,* etc., by means of cords, pulleys and weights, and let the disc find its position of equilibrium. Calculate the moment of each force separately, and attach the proper sign, plus or minus. Take the sum of each kind, and ascertain if the sums are equal, as they should be, according to the principle of moments.

**Resultant of two parallel forces.**—There are two cases to be considered, viz. forces of like sense (Fig. 106 (*a*)) and forces of unlike sense (Fig. 106 (*b*)). The following method is applicable equally to both

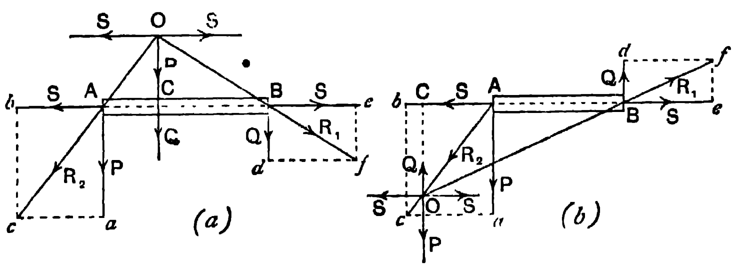


FIG. 106.—Resultant of two parallel forces.

cases, and may be read in reference to both diagrams, which are lettered correspondingly.

For convenience, let the given forces P and Q act at 90° to a rod AB at the points A and B respectively. The equilibrium of the rod will not be disturbed by the application of equal opposite forces S, S, applied in the line of the rod at A and B. By means of the parallelogram of forces *Abca*, find the resultant $R_2$ of P and S acting at A; in the same manner find the resultant $R_1$ of Q and S acting at B. Produce the lines of $R_1$ and $R_2$ until they intersect at O, and let $R_1$ and $R_2$ act at O. Resolve $R_1$ and $R_2$ into components acting at O, and respectively parallel and at right angles to AB; the components parallel to AB will be each equal to S, therefore they balance and need not be considered further. The components at right angles to AB will be equal respectively to P and Q,

and are the only forces remaining. Hence R is equal to their algebraic sum; thus

In Fig. 106 (*a*) $R = P + Q$. ....................................(1)

In Fig. 106 (*b*) $R = P - Q$. ....................................(1*a*)

Let the line of R, which passes through O and is parallel to P and Q, be produced to cut AB, or AB produced, in C. Then the triangles OAC and A*ca* are similar; hence

$$\frac{OC}{CA} = \frac{Aa}{ac} = \frac{Aa}{Ab} = \frac{P}{S}. \quad \text{...............................(2)}$$

Also the triangles OBC and B*fd* are similar, therefore

$$\frac{OC}{CB} = \frac{Bd}{df} = \frac{Bd}{Be} = \frac{Q}{S}. \quad \text{...............................(2a)}$$

Divide (2) by (2*a*), giving

$$\frac{CB}{CA} = \frac{P}{Q}. \quad \text{...............................(3)}$$

This result indicates that the line of the resultant divides the rod into segments inversely proportional to the given forces, internally if the forces are like, and externally if the forces are unlike. It will be noted also that the resultant is always nearer to the larger force; in the case of forces of unlike sense, the resultant has the same sense as the larger force. The equilibrant of P and Q may be obtained by applying to the rod a force equal and opposite to R.

In the case of equal parallel forces of unlike sense, equations (1*a*) and (3) give

$$R = P - P = 0;$$

$$\frac{CB}{CA} = \frac{P}{P} = 1; \quad \therefore CB = CA;$$

the interpretation being that the resultant is a force of zero magnitude applied at infinity. The name **couple** is given to two equal parallel forces of unlike sense; a couple has no resultant force, and hence cannot be balanced by a force. Some properties of couples will be discussed later.

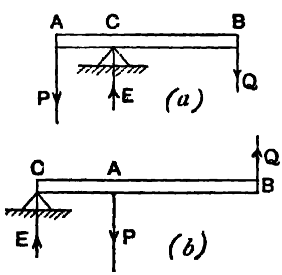

FIG. 107.—Moments of parallel forces.

**Moments of parallel forces.**—The light rods shown in Fig. 107 (*a*) and (*b*) are in equilibrium under the actions of forces P and Q, of like sense in Fig. 107 (*a*) and of unlike sense in Fig. 107 (*b*), together with the equilibrants E, supplied by the reactions of the pivots at C. From equation (3) above, we have

$$\frac{P}{Q} = \frac{BC}{AC},$$

or, $$P \times AC = Q \times BC. \quad \text{...............(1)}$$

This result indicates that the moments of P and Q are equal and opposite, and we may infer that this condition must be fulfilled in order that the rod may not rotate.

In Fig. 108, R is the resultant of P and Q. Take any other point O in the rod, and take moments about O.

$$\text{Moment of } R = R \times OC = R(OA + AC). \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\text{Moment of } P = P \times OA.$$

$$\text{Moment of } Q = Q \times OB = Q(OA + AC + CB).$$

$\therefore$ Moment of $P +$ moment of $Q$

$= (P \times OA) + (Q \times OA) + (Q \times AC) + (Q \times CB)$

$= (P + Q)OA + (Q \times AC) + (P \times AC)$, from (1)

$= (P + Q)OA + (P + Q)AC$

$= R(OA + AC)$

$=$ moment of R.

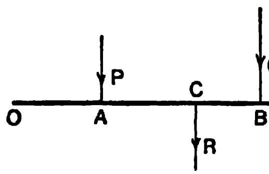


FIG. 108.—Moments of P, Q and R about O.

We may therefore assert that the algebraic sum of the moments of the parallel components of a force about any point is equal to the moment of the resultant.

**Reaction of a pivot.**—In Fig. 109 (*a*) a horizontal rod is in equilibrium under the action of a load W applied at A, another load $P_1$ applied at $B_1$, and a reaction $E_1$ exerted by the pivot at C. It is clear that $E_1$ is equal and opposite to the resultant of $P_1$ and W ; hence

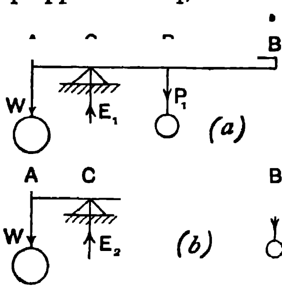


FIG. 109.—Reactions of pivots.

$$E_1 = P_1 + W.$$

In Fig. 109 (*b*) is shown the same rod carrying the same load W at the same place; but now equilibrated by a force $P_2$ applied at $B_2$, and the equilibrant $E_2$ applied by the pivot. As before :

$$E_2 = P_2 + W.$$

In Fig. 109 (*a*) we have

$$P_1 \times B_1C = W \times AC \; ; \quad \therefore \; P_1 = \frac{AC}{B_1C} W.$$

In Fig. 109 (*b*), in the same way :

$$P_2 \times B_2C = W \times AC \; ; \quad \therefore \; P_2 = \frac{AC}{B_2C} W.$$

Since W and AC are the same in both cases, and since $B_2C$ is greater than $B_1C$, $P_2$ is less than $P_1$ ; therefore $E_2$ is less than $E_1$. It will thus be noted that although the general effect is the same in both cases, viz. the rod is in equilibrium, the effects on the pivots are not identical,

nor will be the effects of the loads in producing stresses in the material of the rod.

EXPT. 19.—**Equilibrant of two parallel forces.** Hang a rod AB from a fixed support by means of a cord attached to A (Fig. 110); let the rod hang in front of a vertical board and fix it in its position of equilibrium by means of bradawls at A and B. Apply parallel forces P and Q at C and D respectively, using cords, pulleys and weights $W_1$ and $W_2$. Find the resultant R and its point of application by calculation, and then apply E, equal and opposite to R by means of another cord, pulley and weight $W_3$. Remove the bradawls; if the rod remains unaltered in position, the result shows that the method of calculation has been correct.

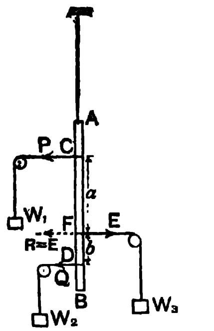

FIG. 110.—Equilibrant of two parallel forces of the same sense.

Repeat the experiment using forces P and Q of unlike sense. Also verify the fact that if P and Q are equal and of opposite sense and act in parallel lines, no single force applied to the rod will preserve equilibrium.

**Resultant of any number of parallel uniplanar forces.**—In Fig. 111 forces P, Q, S, T are applied to a rod at A, B, C, D respectively. The resultant of these forces may be found by successive applications of the methods described on p. 87 for finding the resultant of two parallel forces. O being any convenient point of reference, first find the resultant $R_1$ of P and Q.

$$R_1 = P - Q. \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$R_1x_1 = (P \times OA) - (Q \times OB);$$

$$\therefore\ x_1 = \frac{(P \times OA) - (Q \times OB)}{P - Q}. \quad\ldots(2)$$

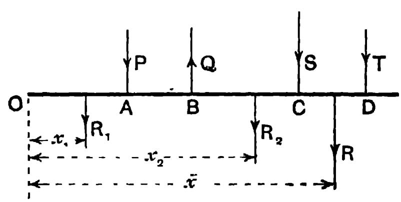

FIG. 111.—Resultant of parallel forces

Now find the resultant $R_2$ of $R_1$ and S.

$$R_2 = R_1 + S = P - Q + S. \quad\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$R_2x_2 = R_1x_1 + (S \times OC) = (P \times OA) - (Q \times OB) + (S \times OC);$$

$$\therefore\ x_2 = \frac{(P \times OA) - (Q \times OB) + (S \times OC)}{P - Q + S}. \quad\ldots\ldots\ldots\ldots(4)$$

In the same manner, find the resultant R of $R_2$ and T; R will then be the resultant of the given forces.

$$R = R_2 + T = P - Q + S + T. \quad\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$R\bar{x} = R_2x_2 + (T \times OD)$$
$$= (P \times OA) - (Q \times OB) + (S \times OC) + (T \times OD) ;$$
$$\therefore \bar{x} = \frac{(P \times OA) - (Q \times OB) + (S \times OC) + (T \times OD)}{P - Q + S + T}. \quad \ldots\ldots\ldots\ldots(6)$$

In this result the numerator is the algebraic sum of the moments about O of the given forces, and may be written $\Sigma Px$. The denominator is the algebraic sum of the given forces, and may be written $\Sigma P$. Hence from (5) and (6), we have

$$R = \Sigma P. \quad \ldots\ldots\ldots\ldots(7)$$
$$\bar{x} = \frac{\Sigma Px}{\Sigma P}. \quad \ldots\ldots\ldots\ldots(8)$$

It is evident that the resultant is parallel to the forces of the given system ; its sense is determined by the sign of the result calculated from (7).

Reversal of R will give the equilibrant of the given system. Should the given forces be in equilibrium, R will be zero, and the algebraic sum of the moments of the given forces will also be zero. Hence the conditions of equilibrium are

$$\Sigma P = 0. \quad \ldots\ldots\ldots\ldots(9)$$
$$\Sigma Px = 0. \quad \ldots\ldots\ldots\ldots(10)$$

These must be satisfied simultaneously.

Should $\Sigma P$ be zero, and $\Sigma Px$ be not zero, then the interpretation is that the system can be reduced to two equal parallel forces of opposite sense, *i.e.* a couple (p. 88). Should $\Sigma P$ have a numerical value and $\Sigma Px$ be zero, then the point O about which moments have been taken lies on the line of the resultant.

**Reactions of a loaded beam.**—The above principles may be applied in the determination of the reactions of the supports of a loaded beam. An example will render the method clear.

EXAMPLE.—A beam AB rests on supports at A and B 16 feet apart and carries loads of 2, 1, 0·75 and 0·5 tons as shown (Fig. 112). Find the reactions P and Q of the supports.

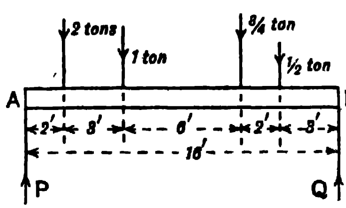


FIG. 112.—Reactions of a beam.

From equation (9) above,

$$\Sigma P = 0 ; \quad \therefore P + Q = 2 + 1 + 0{\cdot}75 + 0{\cdot}5$$
$$= \underline{4{\cdot}25} \text{ tons.}$$

P may be calculated by taking moments about B ; Q has no moment about this point, and will therefore not appear in the calculation.

The sum of the clockwise moments will be equal to the sum of the anticlockwise moments ; hence

$$P \times 16 = (2 \times 14) + (1 \times 11) + (0{\cdot}75 \times 5) + (0{\cdot}5 \times 3) ;$$
$$\therefore P = \underline{2{\cdot}765} \text{ tons.}$$

In the same way, Q may be found by taking moments about A; thus:

$$Q \times 16 = (2 \times 2) + (1 \times 5) + (0{\cdot}75 \times 11) + (0{\cdot}5 \times 13);$$

$$\therefore\ Q = \underline{1{\cdot}484} \text{ tons.}$$

The sum of these calculated values gives

$$P + Q = 2{\cdot}765 + 1{\cdot}484 = \underline{4{\cdot}249} \text{ tons,}$$

a result which agrees with the sum already calculated, viz. 4·25, within the limits of accuracy adopted in the calculations.

EXPT. 20.—**Reactions of a beam.** Suspend a wooden bar from two supports, using spring balances so that the reactions of the supports may be observed (Fig. 113). Prior to placing any loads on the beam, read the

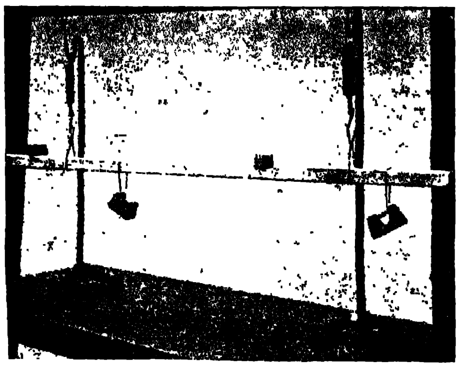

FIG. 113.—Apparatus for determining the reactions of the supports of a beam.

spring balances; let the readings be $P_1$ and $Q_1$ lb. weight respectively. Place some loads on the beam and calculate the reactions, neglecting the weight of the beam. Again read the spring balances, P and Q lb. weight say. The differences $(P - P_1)$ and $(Q - Q_1)$ lb. weight should agree with the values found by calculation.

## Exercises on Chapter VIII

**1.** A bicycle has cranks 7 inches long from the axis to the centre of the pedal. If the rider exerts a constant push of 20 lb. weight vertically throughout the downward stroke, find the turning moment when the crank is at the top, also when it has turned through angles of 30°, 60°, 90°, 120°, 150° and 180° from the top position.

**2.** A wooden disc is capable of turning freely in a vertical plane about a horizontal axis passing through its centre O. Light pegs A and B are driven into one side of the disc; OA = OB = 6 inches and OA and OB are perpendicular to one another. Fine cords are attached to A and B and hang vertically; these cords carry weights of 4 lb. and 2 lb. respectively. Find, and show in a diagram, the positions in which the disc will be in equilibrium.

**3.** Two parallel forces of like sense, one of 8 lb. weight and the other of 6 lb. weight, act on a body in lines 12 inches apart. Find the resultant.

**4.** Answer Question 3 supposing the forces to be of opposite senses.

**5.** A uniform horizontal rod 2 feet long has a weight of 12 lb. hanging from one end, and the rod is pivoted at its centre. Balance has to be restored by means of a weight of 18 lb. Find where it must be placed.

**6.** A rod AB carries bodies weighing 3 lb., 7 lb. and 10 lb. at distances of 2 inches, 9 inches and 15 inches respectively from A. Neglect the weight of the rod and find the point at which the rod must be supported for equilibrium to be possible.

**7.** A beam 12 feet long is supported at its ends and carries a weight of 2·5 tons at a point 4 feet from one end. Neglect the weight of the beam and find the reactions of the supports.

**8.** A lever $3\frac{1}{2}$ feet long is used by a man weighing 150 lb. who can raise unaided a body weighing 300 lb. Find the load he can raise (*a*) applied at the end of the lever, the pivot being 4 inches from this end; (*b*) with the pivot at one end of the lever and the load at 4 inches from this end.

**9.** A light rod AB is 1 metre long and has parallel forces applied at right angles to the rod as follows: At A, 2 kilograms weight; at 15 cm. from A, 4 kilograms weight; at 55 cm. from A, 6 kilograms weight; at B, 8 kilograms weight. Find the resultant of these forces.

**10.** A light horizontal rod AB is 2 feet long and is supported at its ends; the reaction at A acts at 30° to the vertical. Find both reactions if a load of 3 lb. weight is placed on the rod at 8 inches from A.

**11.** A light horizontal rod AB, 3 feet long, is supported at its ends; the reaction at A is vertical. A force of 4 lb. weight is applied at a point C in the rod; AC is 1 foot and the angle between AC and the line of the force is 70°. Find the reactions of both supports.

**12.** A horizontal lever AB is 6 feet long and is pivoted at C; AC is 4 inches. If a load of 400 lb. weight is suspended at A, find the position and magnitude of the weight which must be applied to the lever in order that the reaction of the support shall be a minimum. State the minimum value of the reaction. Neglect the weight of the lever.

**13.** A plank AB, 10 feet long, is hinged at a point 6 feet from a vertical wall and its upper end B rests against the wall. Assume that both hinge and wall are smooth. Find the reactions of the wall and hinge if loads of 40, 60 and 100 lb. weight are hung from points in the plank at distances of 2, 6 and 8 feet respectively from A. Neglect the weight of the plank.

**14.** A bent lever ACB is pivoted at C; the arms AC and BC meet at 120°; AC = 18 inches, BC = 10 inches, and is horizontal. If a load of 150 lb. weight be hung from B, find, by taking moments about C, what horizontal force must be applied at A. Find also the reaction of the pivot at C.

**15.** A beam AB, 40 feet long, is supported at A and at a point 10 feet from B. Loads of 4, 8, 6 and 10 tons weight are applied respectively at points 8, 20, 30 and 40 feet from A. Neglect the weight of the beam and find the reactions of the supports.

**16.** A plank 12 feet long spans an opening between two walls. A man weighing 150 lb. crosses the plank. Find the reactions of the supports of the plank when the man is at distances of 2, 4, 6, 8, 10 feet from one end. Neglect the weight of the plank. Plot a graph showing these distances as abscissae, and the reactions of the left-hand support as ordinates.

**17.** In Question 16, two men A, B, cross the plank from right to left, B keeping at a distance of 4 feet behind A. Each man weighs 150 lb. Find the reactions of the left-hand support when A is at the following distances in feet from it: 12, 10, 8, 6, 4, 2, 0. Neglect the weight of the plank. Plot a graph showing the reactions of the left-hand support as ordinates, and the distances of A from this support as abscissae.

**18.** The seats in a rowing boat are 3 feet apart. The steersman weighs 110 lb. Starting with bow, the weights in lb. of the four oarsmen are as follows: 162, 155, 149, 166. Find the resultant weight in magnitude and position.

**19.** Give the conditions of equilibrium of a body under parallel forces. A thin rod of negligible weight rests horizontally on the hooks of two spring balances suspended 10 inches apart. Two bodies of weight 2 lb. and 3 lb. respectively are hung from the rod, always at a distance of 20 inches apart from each other. How will you suspend these weights so that each spring balance shows the same reading?

**20.** Prove that the algebraic sum of the moments of any two coplanar forces about a point in their plane is equal to the moment of their resultant about that point.

A uniform plane lamina in the form of a regular hexagon ABCDEF is free to rotate in its own plane which is vertical about centre O. Masses 1, 2, 3, 4, 5 and 6 lb. are attached to the vertices A, B, C, D, E and F respectively. Find the inclination of AD to the vertical for which the system is in equilibrium. J.M.B., H.S.C.

**21.** (*a*) Two forces P, Q at A and two forces R, S at B, act on a rigid body of negligible weight. Prove that if P and Q do not balance separately a necessary condition for equilibrium is that their resultant should pass through B.

(*b*) ABCD is a uniform rectangular lamina with sides $AB = 3$ inches, $BC = 4$ inches, smoothly pivoted at B. It is acted on by forces 9 lb., 12 lb., 30 lb. along CD, DA, AC respectively. Prove that the rectangle is in equilibrium and find completely the reaction at the pivot.

J.M.B., H.S.C.

# CHAPTER IX

## CENTRE OF PARALLEL FORCES. CENTRE OF GRAVITY

**Centre of parallel forces.**—In Fig. 114, AB is a rod having forces P and Q applied at the ends in lines making 90° with the rod. The resultant R of P and Q divides the rods into segments given by

$$P : Q = BC : AC \text{ (p. 88).} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Without altering the magnitudes of P and Q, let their lines be kept parallel and rotated into new positions P′ and Q′. Through C draw DCE perpendicular to P′ and Q′. The resultant R′ of P′ and Q′ divides DE into segments inversely proportional to P′ and Q′. It is evident that the triangles ACD and BCE are similar; hence

$$EC : DC = BC : AC = P : Q$$

from (1). It therefore follows that R′ passes through the same point C. This point is called the centre of the parallel forces P and Q.

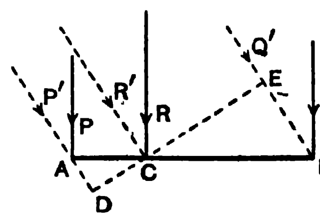

FIG. 114.—Centre of parallel forces.

If several parallel forces act on the rod, it may be shown easily that their resultant always passes through the same point so long as the forces remain unaltered in magnitude.

**Centre of gravity.**—Every particle in a body possesses weight; hence the gravitational effort exerted on any body consists of a large number of forces directed towards the centre of the earth. These forces are sensibly parallel in the case of any body of moderate size. It is not possible to alter the directions of the forces of gravitation to any appreciable extent, but a similar effect may be obtained by inclining the body. The weights of all the particles still act in vertical lines, but their directions will be altered in relation to any fixed line AB in the body (Fig. 115 (*a*) and (*b*)).

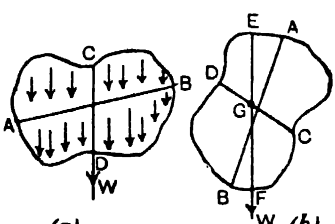

FIG. 115.—Centre of gravity.

Let W be the resultant weight of the body, and let its line of action be marked at CD on the body in Fig. 115 (*a*), and to be marked again as EF in Fig. 115 (*b*). CD and EF intersect at G, and it is clear

from what has been said that W will pass through G whatever may be the position of the body. G is the centre of the weights of the particles composing the body, and is called the **centre of gravity.**

In taking moments of the forces acting on a body, the simplest way of dealing with the total weight of the body is to imagine it to be applied as a vertical force concentrated at the centre of gravity of the body. The centre of gravity of a body may be defined as **that point at which the total weight of the body may be imagined to be concentrated without thereby altering the gravitational effect on the body.**

EXAMPLE.—A uniform beam AB weighs 1·5 tons, and has its centre of gravity at the middle of its length. The beam is 16 feet long, and is supported at the end A and at a point C 4 feet from the end B (Fig. 116). Loads of 2 and 3 tons weight are applied at 3 feet from A and at B respectively. Find the reactions of the supports.

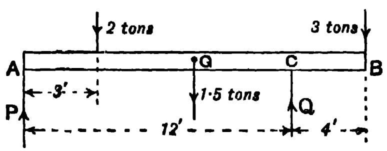


FIG. 116.

The centre of gravity G is at a distance of 8 feet from A; apply the weight of the beam, 1·5 tons weight, at G, and take moments about C in order to find P.

$$(P \times 12) + (3 \times 4) = (2 \times 9) + (1{\cdot}5 \times 4).$$

$$P = \frac{18 + 6 - 12}{12} = \frac{12}{12}$$

$$= \underline{1} \text{ ton weight.}$$

To find Q take moments about A.

$$Q \times 12 = (3 \times 16) + (1{\cdot}5 \times 8) + (2 \times 3).$$

$$Q = \frac{48 + 12 + 6}{12} = \frac{66}{12}$$

$$= \underline{5{\cdot}5} \text{ tons weight.}$$

Check: $P + Q = 1 + 5{\cdot}5 = 6{\cdot}5$ tons weight
$=$ total load on the supports.

**Some simple cases of centre of gravity.**—The position of the centre of gravity in certain simple cases may be located by inspection. Thus, for a slender straight rod or wire of uniform cross-section, the centre of gravity G lies at the middle of the length. In a thin uniform square or rectangular plate, G lies at the intersection of the diagonals; a thin uniform circular plate has G at its geometrical centre.

A parallelogram made of a thin uniform sheet may be imagined to be built up of thin uniform rods arranged parallel to AB (Fig. 117); the centre of gravity of each rod lies at the middle of length of the rod, hence all their centres of gravity lie in HK, which bisects AB and CD. The centre of gravity of the parallelogram therefore lies in HK. Similarly,

the plate may be imagined to be constructed of thin rods lying parallel to AD, and the centres of gravity of all these rods, and therefore the centre of gravity of the parallelogram, will lie in EF, which bisects AD and BC. Hence G lies at the point of intersection of HK and EF; it is evident that the diagonals AC and BD intersect at G.

A thin triangular plate may be treated in a similar manner (Fig. 118). First take strips parallel to AB, when it is clear that the centre

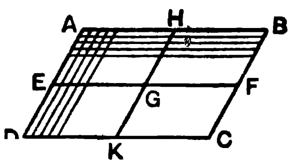


FIG. 117.—Centre of gravity of a parallelogram.

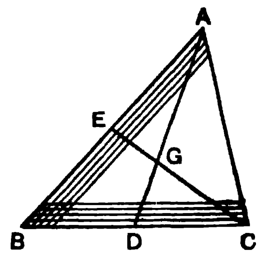


FIG. 118.—Centre of gravity of triangle.

of gravity of the plate lies in CE, which bisects AB and also bisects all the strips parallel to AB. Then take strips parallel to BC; AD bisects BC and also all the strips parallel to BC, and therefore contains the centre of gravity. Hence G lies at the intersection of CE and AD.

Let DE be joined in Fig. 118, then the triangles BED and BAC are similar, since DE is parallel to AC. Therefore $DE = \frac{1}{2}AC$. Also the triangles DEG and ACG are similar; hence

$$\frac{DG}{AG} = \frac{DE}{AC} = \frac{\frac{1}{2}AC}{AC} = \frac{1}{2};$$

$$\therefore\ DG = \tfrac{1}{2}AG = \tfrac{1}{3}AD.$$

We have therefore the rule that the centre of gravity of a thin triangular sheet lies on the line drawn from the centre of a side to the opposite corner, and one-third of its length from the side.

Any uniform prismatic bar has its centre of gravity in its axis at the middle of its length. The centre of gravity of a uniform sphere lies at its geometrical centre. A solid cone or pyramid has the centre of gravity on the line joining the centre of the base to the apex, and one-quarter of its length from the base. A cone or pyramid open at the base, and made of a thin sheet bent to shape, has its centre of gravity on the line joining the centre of the base to the apex, and one-third of its length from the base.

**Method of calculating the position of the centre of gravity.**—The problem of finding a line which contains the centre of gravity of a body is identical with that of finding the line of action of the resultant weight of the body, having been given the weights of the separate particles of which the body is composed.

Fig. 119 shows a thin uniform sheet in the plane of the paper; OX and OY are horizontal and vertical axes of reference. Let $x_1$, $y_1$ be the coordinates of a particle at P; let the weight of the particle be $w_1$, and describe similarly all the other particles of the body. Then

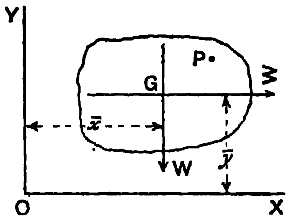


FIG. 119.—Centre of gravity of a sheet.

Resultant weight of the body

$$= W = w_1 + w_2 + w_3 + \text{etc.}$$
$$= \Sigma w. \qquad (1)$$

Take moments about O, and let $\bar{x}$ be the horizontal distance between the line of W and OY; then

$$W\bar{x} = w_1x_1 + w_2x_2 + w_3x_3 + \text{etc.}$$
$$\therefore \bar{x} = \frac{\Sigma wx}{W}. \qquad (2)$$

Now turn the figure until OX becomes vertical, and again take moments about O; let the distance between the line of W and OX be $y$, then

$$W\bar{y} = w_1y_1 + w_2y_2 + w_3y_3 + \text{etc.}$$
$$\therefore \bar{y} = \frac{\Sigma wy}{W}. \qquad (3)$$

Draw a line parallel to OX and at a distance $\bar{y}$ from it; draw another line parallel to OY and at a distance $\bar{x}$ from it; the centre of gravity G lies at the intersection of these lines.

Generally the sheet under consideration may be cut into portions for each of which the weights $w_1$, $w_2$, $w_3$, etc., may be calculated, and the coordinates of the centres of gravity $(x_1y_1)$, $(x_2y_2)$, etc., may be written down by inspection.

EXAMPLE 1.—Find the centre of gravity of the thin uniform plate shown in Fig. 120.

Take axes OX and OY as shown and let the weight of the plate per square inch of surface be $w$. For convenience of calculation the plate is divided into three rectangles as shown, the respective centres of gravity being $G_1$, $G_2$ and $G_3$. Taking moments about OY, we have

$$w\{(6 \times 1) + (8 \times 1) + (3 \times 1)\}\bar{x}$$
$$= w(6 \times 1 \times 3) + w(8 \times 1 \times \tfrac{1}{2}) + w(3 \times 1 \times 1\tfrac{1}{2}),$$
$$\bar{x} = \frac{26 \cdot 5}{17} = \underline{1 \cdot 56} \text{ inches.}$$

Again, taking moments about OX, we have

$$17\bar{y} = (6 \times 1 \times 9\tfrac{1}{2}) + (8 \times 1 \times 5) + (3 \times 1 \times \tfrac{1}{2}),$$
$$\bar{y} = \frac{98 \cdot 5}{17} = \underline{5 \cdot 8} \text{ inches.}$$

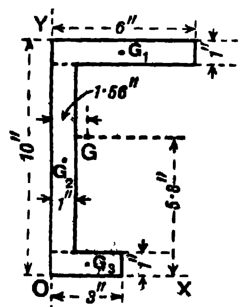


FIG. 120.

EXAMPLE 2.—A circular plate (Fig. 121) 12 inches diameter has a hole 3 inches diameter. The distance between the centre A of the plate and the centre B of the hole is 2 inches. Find the centre of gravity.

Take AB produced as OX, and take OY tangential to the circumference of the plate. It is evident that G lies in OX. Taking moments about OY, we may say that the moment of the plate as made is equal to that of the complete disc diminished by the moment of the material removed in cutting out the hole. Let $w$ be the weight per square inch of surface, D the diameter of the plate, and $d$ that of the hole. Then

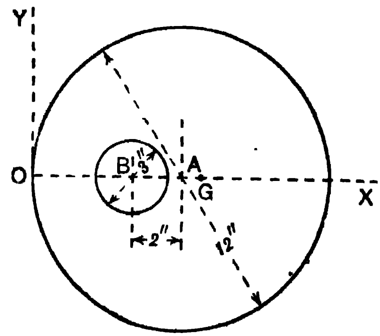

FIG. 121.

$$\text{Weight of the complete disc} = w\frac{\pi D^2}{4}.$$

$$\text{Weight of the piece cut out} = w\frac{\pi d^2}{4}.$$

$$\text{Weight of the plate as made} = w\left(\frac{\pi D^2}{4} - \frac{\pi d^2}{4}\right) = \frac{w\pi}{4}(D^2 - d^2).$$

Take moments about OY, and let $OG = \bar{x}$,

$$\frac{w\pi}{4}(D^2 - d^2)\bar{x} = \left(w\frac{\pi D^2}{4} \times 6\right) - \left(w\frac{\pi d^2}{4} \times 4\right),$$

$$(D^2 - d^2)\bar{x} = 6D^2 - 4d^2,$$

$$\bar{x} = \frac{6D^2 - 4d^2}{D^2 - d^2} = \frac{828}{135} = \underline{6 \cdot 13} \text{ inches}.$$

EXAMPLE 3.—A notice board 3 feet broad by 2 feet high, made of timber 1 inch thick, is nailed to a post 7 feet high, made of the same kind of timber, 3 inches by 3 inches (Fig. 122). Find the centre of gravity.

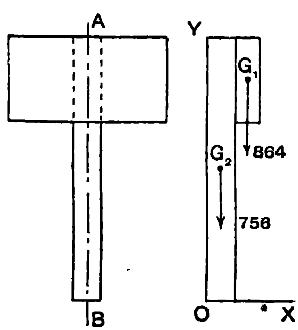

FIG. 122.

The volumes of the post and board are proportional to the weights, and may be used instead of the weights.

The weight of the board is proportional to $(36 \times 24 \times 1) = 864$.

The weight of the post is proportional to $(84 \times 3 \times 3) = 756$.

The total weight is proportional to

$$(864 + 756) = 1620.$$

The vertical plane of which AB is the trace (Fig. 122) contains the centres of gravity of both board and post, and therefore contains the centre of gravity of the whole.

*Take moments about O.*

$$1620\bar{x} = (864 \times 3 \cdot 5) + (756 \times 1 \cdot 5)$$

$$\bar{x} = \frac{3024 + 1134}{1620} = \underline{2 \cdot 566} \text{ inches.}$$

Also,

$$1620\bar{y} = (864 \times 72) + (756 \times 42).$$

$$\bar{y} = \frac{62,208 + 31,752}{1620} = \underline{58} \text{ inches.}$$

Hence the centre of gravity lies in the vertical plane AB, at a height of 58 inches and at 2·566 inches from the back of the post.

EXAMPLE 4.—Bodies having weights of $w_1$, $w_2$, $w_3$, $w_4$ are placed in order at the corners A, B, C, D of a square (Fig. 123). Find the centre of gravity.

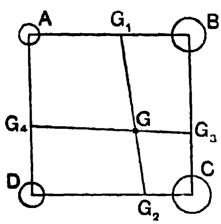


FIG. 123.

First find the centre of gravity $G_1$ of $w_1$ and $w_2$; $G_1$ falls in AB and divides AB in the proportion

$$\frac{AG_1}{BG_1} = \frac{w_2}{w_1}.$$

In the same way, the centre of gravity of $w_3$ and $w_4$ falls in CD, and divides CD in the proportion

$$\frac{CG_2}{DG_2} = \frac{w_4}{w_3}.$$

The centre of gravity of the whole system lies in $G_1G_2$.

The centre of gravity of $G_3$ of $w_2$ and $w_3$ divides BC in the proportion

$$\frac{BG_3}{CG_3} = \frac{w_3}{w_2}.$$

Also the centre of gravity of $w_1$ and $w_4$ divides AD in the proportion

$$\frac{AG_4}{DG_4} = \frac{w_4}{w_1}.$$

The centre of gravity G of the whole system lies in $G_3G_4$. Therefore G lies at the intersection of $G_1G_2$ and $G_3G_4$. The completion of the problem may be carried out on a drawing made carefully to scale.

EXAMPLE 5.—Equal weights $w_1$, $w_2$ and $w_3$ are placed at the corners of any triangle ABC (Fig. 124). Find the centre of gravity.

The centre of gravity $G_1$ of $w_1$ and $w_2$ bisects AB, and the centre of gravity of the system lies in $CG_1$. Also the centre of gravity $G_2$ of $w_2$ and $w_3$ bisects BC, and the centre of gravity G of the system lies in $AG_2$. Hence G is at the intersection of two lines drawn from the centres of two sides to the opposite corners of the triangle, and therefore coincides with the centre of gravity of a thin sheet having the same shape as the triangle.

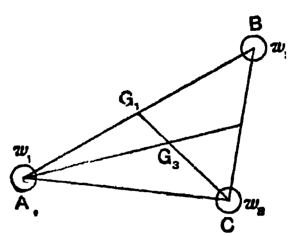


FIG. 124.

**Example 6.**—In Fig. 125 (*a*), ABCD is a thin sheet; AB is parallel to CD. Find the centre of gravity. (This is a case which occurs often in practice.)

Imagine the sheet to be divided into narrow strips parallel to AB. The centre of gravity of each strip will lie in EF, a straight line which bisects both AB and CD, and therefore bisects each strip. Join DE and CE, thus dividing the sheet into three triangles ADE, BCE and DEC. Since these triangles are all of the same height, their areas and hence their weights are proportional to their bases; thus

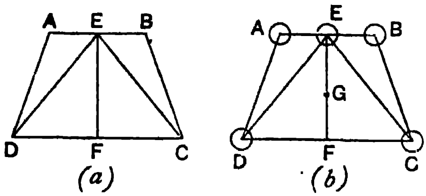


Fig. 125.—A frequent case of centre of gravity.

$$\text{Weight of } \triangle ADE : \text{weight of } \triangle BCE : \text{weight of } \triangle DEC = \tfrac{1}{2}AB : \tfrac{1}{2}AB : DC.$$

Making use of the proposition discussed in Example 5 above, the weight of each triangle may be divided into three equal parts and one part placed at each corner of the triangle without altering the position of the centre of gravity of the triangle. The equivalent system of weights will be as follows (Fig. 125 (*b*)).

$$\text{At A, } \tfrac{1}{6}AB\text{; at B, } \tfrac{1}{6}AB\text{; at E, } (\tfrac{2}{6}AB + \tfrac{1}{3}DC)\text{; at C, } (\tfrac{1}{6}AB + \tfrac{1}{3}DC)\text{;}$$
$$\text{at D, } (\tfrac{1}{6}AB + \tfrac{1}{3}DC).$$

The centre of gravity of the weights at A, E and B lies at E, and the resultant of these weights is $(\tfrac{2}{6}AB + \tfrac{2}{6}AB + \tfrac{1}{3}DC) = \tfrac{4}{6}AB + \tfrac{1}{3}DC$. The centre of gravity of the weights at C and D is at F, and the resultant of these weights is $(\tfrac{2}{6}AB + \tfrac{2}{3}DC)$. The centre of gravity G of the whole system divides EF in the proportion

$$\frac{FG}{EG} = \frac{\tfrac{4}{6}AB + \tfrac{1}{3}DC}{\tfrac{2}{6}AB + \tfrac{2}{3}DC} = \frac{2AB + DC}{AB + 2DC}. \quad \text{.........................(1)}$$

The following graphical method (Fig. 126) is useful in this case: Draw EF as before; produce AB and CD to H and K respectively, making BH = CD, and DK = AB. Join HK cutting EF in O. The triangles EOH and FOK are similar, hence

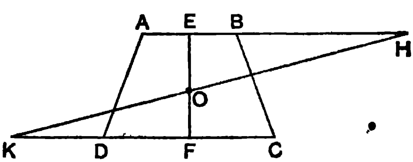


Fig. 126.

$$\frac{FO}{EO} = \frac{FK}{EH} = \frac{DK + FD}{EB + BH}$$
$$= \frac{AB + \tfrac{1}{2}DC}{\tfrac{1}{2}AB + DC}$$
$$= \frac{2AB + DC}{AB + 2DC}. \quad \text{..........(2)}$$

As this result is identical with that found in (1) for the position of G, it follows that O and G coincide. Hence Fig. 126 provides a purely graphical method of finding the centre of gravity of the sheet.

**States of equilibrium of a body.**—When a body is at rest under the action of a system of forces, the equilibrium is **stable** or **unstable** according as the body returns, or fails to return to its original position after being disturbed slightly. The equilibrium is **neutral** if the body remains at rest in any position. When a body is at rest under the action of gravity and given supporting forces, the state of equilibrium depends, among other conditions, upon the situation of the centre of gravity.

A cone may assume any of the three states of equilibrium. In Fig. 127 (*a*) the cone is resting with its base on a horizontal table; if disturbed slightly (Fig. 127 (*b*)), the tendency of W, acting through the

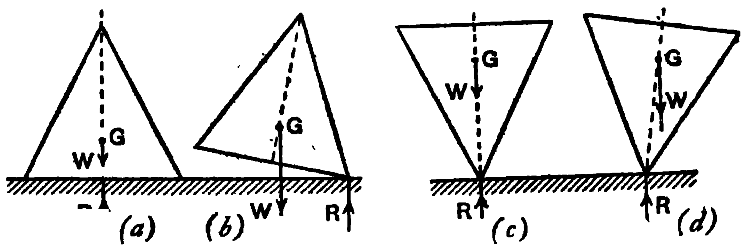

FIG. 127.—Stable and unstable equilibrium.

centre of gravity G, with the reaction R of the table, is to restore the cone to its original position. The equilibrium in Fig. 127 (*a*) is therefore stable. In Fig. 127 (*c*) the cone is in equilibrium when resting on its apex; the slightest disturbance (Fig. 127 (*d*)) will bring W and R into parallel lines, and they then conspire to upset the cone. The equilibrium in Fig. 127 (*c*) is therefore unstable. In Fig. 128 the cone is lying on its side on the horizontal table; in this case it is impossible for W and R to act otherwise than in the same vertical line, no matter how the cone may be turned while still lying on its side. Hence the equilibrium is neutral.

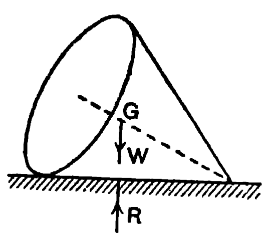

FIG. 128.—Neutral equilibrium.

A sphere resting on a horizontal table is in neutral equilibrium, provided the centre of gravity coincides with the geometrical centre. A cylinder resting with its curved surface on a horizontal table is in neutral equilibrium, so far as disturbance by rolling is concerned, provided the centre of gravity lies in the axis of the cylinder.

In Fig. 129 (*a*) a rectangular block rests on a horizontal plank, one end of which can be raised. The vertical through G falls within the surfaces in contact *ao*, and the equilibrium is stable under the action of W and the reaction R. It is impossible that R can act outside of *ab*; hence stable equilibrium just ceases to be possible when the plank is

inclined to such an angle that the vertical through G passes through *a* (Fig. 129 (*b*)). It is understood that means are provided at *a* to prevent slipping of the block. If the plank be inclined at a steeper angle (Fig. 129 (*c*)), R and W conspire to upset the block.

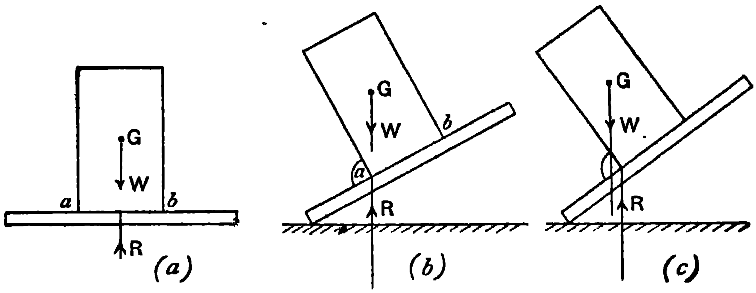


FIG. 129.—Stability of a block on an inclined plane.

It will be noticed that when a body is in a position of stable equilibrium, a disturbance by tilting has the effect of raising the centre of gravity. Further, if a body is capable of moving under the action of gravitational effort, it will always move in such a way as to bring the centre of gravity into a lower position. A position of stable equilibrium will be attained when (as in a pendulum) the centre of gravity has reached the lowest possible position.

In Fig. 130 (*a*) is shown a sphere having its geometrical centre at C and its centre of gravity at G. This displacement of the centre of gravity may be produced either by introducing a heavy plug into the lower hemisphere, or by cutting a slice off the top of the sphere. The sphere rests on a horizontal table, and will be in equilibrium when C and G are both in the same vertical. The reaction R and the weight W are then in the same straight line. If slightly disturbed (Fig. 130 (*b*)), R and W conspire to restore the sphere to the original position, which is therefore a position of stable equilibrium.

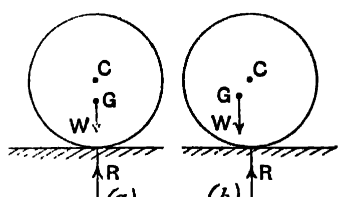


FIG. 130.—Stability of a loaded sphere.

**Graphical methods for finding the centre of gravity of a thin sheet.**—The sheet *abcd* (Fig. 131) is quadrilateral, and is drawn carefully to scale. Divide the sheet into two triangles by joining *bd*. Bisect *bd* in *e*; join *ae* and *ce*; make $ec_1 = \frac{1}{3}ae$, and $ec_2 = \frac{1}{3}ec$; then $c_1$ and $c_2$ are the centres of gravity of the triangles *abd* and *cbd* respectively. Join $c_1c_2$; the centre of gravity of the sheet lies in $c_1c_2$. Again divide the sheet by joining *ac*, and in the same way find the centres of gravity $c_3$ and $c_4$ of the triangles *abc* and *adc*. The centre of gravity G then lies at the intersection of $c_1c_2$ and $c_3c_4$.

In Fig. 132 (*a*) the sheet has a curved outline. Take as reference axes OX touching the outline at its lowest point, and OY at 90° to OX.

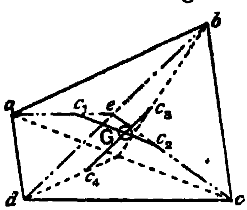


FIG. 131.—Centre of gravity by construction.

Draw AB parallel to OX and touching the outline at its highest point. Let $h$ be the perpendicular distance between OX and AB. CD is a very narrow strip parallel to OX and at a distance $y$ from it, and has a breadth $\delta y$. The area of the strip is proportional to its weight and is equal to $CD \times \delta y$. The moment of this about OX is $CD \times \delta y \times y$. Draw CE and DF perpendicular to AB; join EO and FO cutting CD in H and K. In the similar triangles OHK and OEF, we have

$$HK : EF = y : h;$$

$$\therefore \; HK \times h = EF \times y = CD \times y.$$

Multiply each side by $\delta y$, giving

$$HK \times \delta y \times h = CD \times \delta y \times y.$$

This equation indicates that the product of the area of the strip HK and $h$ is equal to the moment about OX of the strip CD. A similar result may be found for any other strip; hence, if a number of points, such as H and K, be found and a curve drawn through them, the area enclosed by this curve (shown shaded in Fig. 132 (*a*), when multiplied

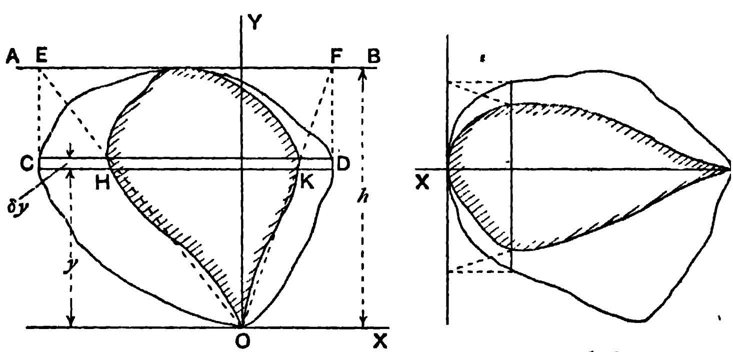


FIG. 132.—Graphical method for finding the centre of gravity of a sheet.

by the constant $h$, will give the total moment about OX of all the strips resembling CD into which the sheet may be divided. Let $A_1$ and $A_2$ be the areas of the given sheet and the shaded curve respectively (these areas can be found by means of a planimeter), and let $\bar{y}$ be the distance of the centre of gravity from OX, then

$$A_2 h = A_1 \bar{y},$$

or,

$$\bar{y} = \frac{A_2}{A_1} h.$$

Take another pair of axes, OX and OY (Fig. 132 (*b*)), and carry out the same process, thus determining the distance $\bar{x}$ of the centre of gravity from OY. The coordinates $\bar{x}$ and $\bar{y}$ of the centre of gravity have now been found.

EXAMPLE.—Apply the above method to find the centre of gravity of a thin semicircular sheet (Fig. 133). The diameter AB is 5 inches.

The centre of gravity of the sheet lies in OC, which is a radius drawn perpendicular to AB, hence $\bar{y}$ alone need be determined by the graphical method, which is shown in Fig. 133.

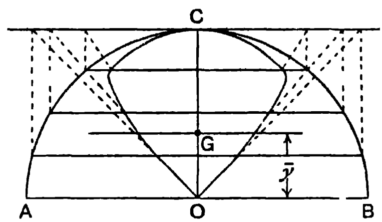

FIG. 133.—Centre of gravity of a semicircular sheet.

The following measurements were obtained by means of a planimeter:

The semicircular area

$$A_1 = 9{\cdot}82 \text{ sq. ins.}$$

Also, $$A_2 = 4{\cdot}12 \text{ sq. ins.}$$

$$\therefore\ \bar{y} = \frac{A_2}{A_1} h = \frac{4{\cdot}12}{9{\cdot}82} \times 2{\cdot}5$$
$$= 1{\cdot}05 \text{ inches.}$$

It is known that the centre of gravity of a semicircular sheet lies at a distance $4r/3\pi$ from AB (Fig. 133). Using this expression in order to check the above result, we obtain

$$\bar{y} = \frac{4 \times 2{\cdot}5}{3 \times \pi} = \underline{1{\cdot}06} \text{ inches.}$$

**Positions of equilibrium.**—If a body is suspended freely from a fixed point, the position in which it will hang in equilibrium is such that the centre of gravity falls in the vertical passing through the fixed point.

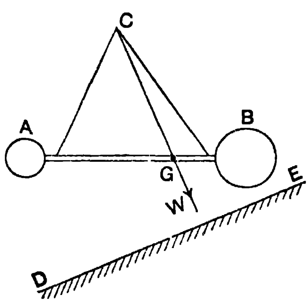

FIG. 134.—A loaded rod.

EXAMPLE 1.—A loaded rod AB (Fig. 134) is suspended by means of two cords from a fixed point C. Find its position of equilibrium.

The centre of gravity, G, of the loaded rod is first found by application of the foregoing methods. Join CG and produce it. The weight W of the system acts in CG; hence CG is vertical. Draw DE perpendicular to CG and turn the paper round until DE is horizontal. The system will then be in its position of equilibrium.

EXAMPLE 2.—A body ABC (Fig. 135 (*a*)), of weight W, is suspended freely from C, and AB is then horizontal. The centre of gravity G bisects AB, and CG is at 90° to AB. Find the angle which AB makes with the horizontal when a body having a weight $w$ is attached at B (Fig. 135 (*b*)).

The centre of gravity G′ now lies in GB, and divides it in the ratio

$$\frac{GG'}{BG'} = \frac{w}{W};$$

$$\therefore \frac{GG'}{GG' + BG'} = \frac{w}{W + w};$$

$$\therefore GG' = \left(\frac{w}{W + w}\right) GB. \qquad \text{.........................(1)}$$

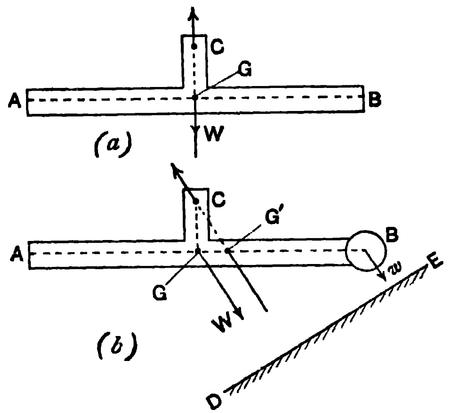


FIG. 135.—Position of equilibrium of a loaded body.

Join CG′ and produce it; draw DE at 90° to CG′, then when the paper is turned so that DE is horizontal, the system is in its position of equilibrium. The angle which AB then makes with the horizontal will be equal to the angle GCG′; let this angle be $\theta$, then

$$\tan \theta = \frac{GG'}{CG} = \left(\frac{w}{W + w}\right) \frac{GB}{CG}$$

(from (1)). ...(2)

From this result we see that if CG is diminished the angle $\theta$ becomes larger; if C and G coincide, CG is zero, $\tan \theta$ is then infinity and the system would hang in equilibrium with CB vertical.

EXPT. 21.—**Centre of gravity of sheets.** The centre of gravity of a thin sheet may be found by hanging it from a fixed support by means of a cord AB (Fig. 136); the cord extends downwards and has a small weight W, thus serving as a plumb-line. Mark the direction AC on the sheet, and then repeat the operation by hanging the sheet from D, marking the new vertical DE. G will be the point of intersection of AC and DE. Carry out this experiment for the sheets of metal or millboard supplied.

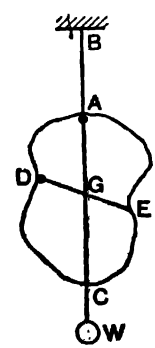


FIG. 136.—Centre of gravity of a sheet by experiment.

EXPT. 22.—**Centre of gravity of a solid body.** Arrange the body so that it is supported on knife edges placed on the pans of balances (Fig. 137). Find the weights $W_1$ and $W_2$ required to restore the balances to equilibrium; these give the reactions of the supports. Measure AB, the distance between the knife edges. Let G be the centre of gravity, then

$$\frac{AG}{BG} = \frac{W_2}{W_1};$$

$$\therefore \frac{AG}{AG+BG} = \frac{W_2}{W_1+W_2};$$

$$\therefore AG = \frac{W_2}{W} . AB,$$

where $W = W_1 + W_2$ is the weight of the body.

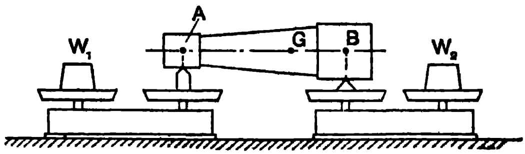


FIG. 137.—Experimental determination of the centre of gravity of a body.

**The common balance.**—In the outline drawing given in Fig. 138, the beam AB is capable of turning freely about a knife edge at C, and its centre of gravity is at G. Scale-pans are hung from knife edges at A and B. If the scale-pans be removed, the beam will remain at rest with G in the vertical passing through C. AB intersects CG at 90° at D, and is therefore horizontal. For the balance to be true, AB must remain horizontal when the scale-pans are hung from the beam, and also when bodies of equal weight are placed in the pans. These conditions will be complied with if AD and BD are equal, and if the scale-pans are of equal weights.

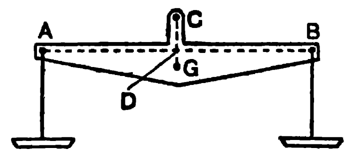


FIG. 138.—Principle of the common balance.

EXAMPLE.—The beam of a balance is shown in Fig. 139. Unequal weights, $W_1$ and $W_2$, $W_2$ being the greater, have been placed in the pans. Determine the angle $\alpha$ which AB now makes with the horizontal.

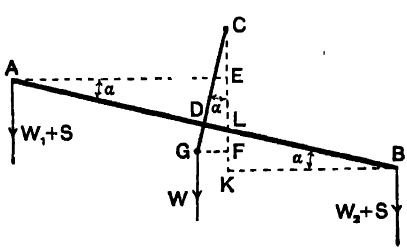


FIG. 139.—Unequally loaded balance.

Let S be the weight of each scale-pan, W the weight of the beam and any attachments fixed rigidly to it. Let $CD = a$, $CG = b$, and $AD = DB = c$. It is evident that CG is inclined at $\alpha$ to the vertical. Draw AE, BK, GF horizontally to meet the vertical CK; let this vertical cut AB in L. Take moments about C, giving

$$(W_1 + S)AE + W . GF = (W_2 + S)BK,$$

$$(W_1 + S)AL \cos\alpha + W . CG . \sin\alpha = (W_2 + S)BL . \cos\alpha,$$

$$(W_1 + S)(AD + DL)\cos\alpha + W . b . \sin\alpha = (W_2 + S)(BD - DL)\cos\alpha,$$

$$(W_1+S)(c+a\tan\alpha)\cos\alpha+W\,.\,b\,.\sin\alpha=(W_2+S)(c-a\tan\alpha)\cos\alpha,$$
$$(W_1+S)(c+a\tan\alpha)+Wb\tan\alpha=(W_2+S)(c-a\tan\alpha),$$
$$(aW_1+aS+Wb+aW_2+aS)\tan\alpha=W_2c+Sc-W_1c-Sc,$$
$$\therefore\ \tan\alpha=\frac{(W_2-W_1)c}{Wb+(W_1+W_2+2S)a}.$$

The magnitude of the angle $\alpha$ for a given difference in weights $(W_2-W_1)$ may be taken as a measure of the **sensitiveness** of a balance. The factors influencing the magnitude of $\alpha$ are given in the formula found above for $\tan\alpha$. Increase in the lengths of the arms $AD=DB=c$ (Fig. 139) will increase $\alpha$, and hence will increase the sensitiveness. The sensitiveness is diminished by increasing the product $Wb$; hence the weight W of the beam should be reduced to the minimum consistent with sufficient rigidity; greater sensitiveness can be obtained by diminishing $CG=b$ (Fig. 139). Diminishing $CD=a$ will increase the sensitiveness, and in many laboratory balances C and D coincide. If G also coincides with D, the result will be a loss of **stability**, since the beam would then be capable of resting in equilibrium at any angle to the horizontal. In a sensitive balance, G falls a little below D, and C may coincide with D. The sensitiveness is diminished by an increase in $(W_1+W_2+2S)$; hence balances intended for delicate work are unsuitable for weighing heavy bodies, and the scale-pans of delicate balances should be light. In order to understand how these principles are applied, the student should examine the parts of a delicate balance.

**Truth of a balance.**—A true balance having equal masses in the pans will vibrate through equal angles above and below the horizontal. The truth may be tested by placing masses in the pans until this condition is fulfilled; the masses are then interchanged, when equal angles will again be observed if the balance is true.

Referring to Fig. 138, let the arms AD and BD be unequal, and let the balance be so constructed that AB remains horizontal, or vibrates through equal angles above and below the horizontal when the scale-pans are empty. Let a body having a true weight W be placed in the left-hand pan, and let it be balanced by a weight P in the other pan. Now place W in the right-hand pan, and let Q be the weight required in order to equilibrate. Take moments in each case about C (Fig. 138).

$$W\times AD=P\times BD. \quad\quad (1)$$
$$W\times BD=Q\times AD. \quad\quad (2)$$

Taking products, we have

$$W^2\times AD\times BD=P\times Q\times BD\times AD;$$
$$\therefore\ W=\sqrt{PQ}. \quad\quad (3)$$

Thus the true weight is the geometrical mean of the false weights P and Q. Had the arithmetical mean $\frac{1}{2}(P+Q)$ been taken as the true weight, the result would be greater than W.

## Exercises on Chapter IX

1. A uniform beam, 12 feet long, weighs 500 lb., and carries a load of 6000 lb. distributed uniformly along its left-hand half. If the beam is supported at its ends, find the reactions of the supports.

2. The jib of a derrick crane (p. 80) is 30 feet long and weighs 800 lb.; the centre of gravity is 12 feet from the lower end. The post and tie of the crane measure 16 feet and 20 feet respectively. Find the pull in the tie necessary to support the jib.

3. A ladder AB, 20 feet long, weighs 90 lb., and its centre of gravity is 8 feet from A. The ladder is carried in a horizontal position by two men, one being at A. A bag of tools weighing 60 lb. is slung at a point 12 feet from A. Find where the second man must be situated if the men share the total load equally between them.

4. A plate of iron of uniform thickness is cut to the shape of a triangle having sides $AB = 2$ feet, $BC = 3$ feet, $CA = 4$ feet. If the plate weighs 50 lb. and lies on a horizontal floor, find what vertical force, applied at one corner, will just lift that corner.

5. A thin plate is cut to the shape shown in Fig. 140. Find its centre of gravity.

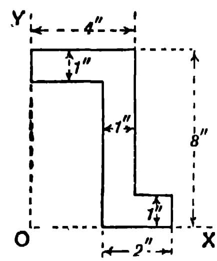

FIG. 140.

6. Draw full size a quadrilateral ABCD;

$AB = 4$ inches, $BC = 2\frac{1}{2}$ inches,

$CD = 3\frac{1}{2}$ inches, $DA = 3\frac{3}{4}$ inches;

diagonal $AC = 4\frac{1}{4}$ inches. The figure represents a thin plate; find its centre of gravity. If the plate weighs 2 lb. and lies on a table, what vertical force would just lift the corner D?

7. A thin plate is cut to the shape of an equilateral triangle of 18 inches side. From one corner is cut off an equilateral triangle of 6 inches side. Find the centre of gravity of the remainder of the plate.

8. A thin circular plate 20 inches diameter has two radii drawn on it meeting at 90°. A circular hole 6 inches diameter has its centre on one radius at a distance of 3 inches from the edge of the plate; another circular hole 4 inches diameter has its centre on the other radius at a distance of 2 inches from the edge of the plate. Find the centre of gravity of the plate after the holes have been cut.

9. A rectangular iron plate (Fig. 141) measures 14 inches by 8 inches by $1\frac{1}{2}$ inches thick. A hole 2 inches diameter is bored through the plate, its centre being 5 inches from one edge and 2 inches from the adjacent edge of the plate. An iron rod 2 inches diameter and 20 inches long is pushed into the hole, its end being flush with the face of the plate. Find the centre of gravity of the arrangement.

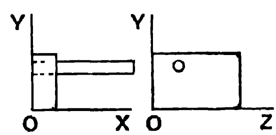

FIG. 141.

10. A plank of uniform cross-section weighs 400 lb. and is 12 feet long. It is supported at one end and at a point 3 feet from the other end. Find the reactions of the supports. Find also the greatest load which can be placed at the end which overhangs without tilting the plank; when this load is applied, what are the reactions of the supports?

11. *A rectangular block of stone stands on one end on a horizontal* surface. The block measures 4 feet high, 2 feet broad and 2 feet thick. If stone weighs 150 lb. per cubic foot, find what horizontal force applied at the top of the block at the centre of one edge, will just produce tilting. Slipping is prevented.

12. The block given in Question 11 rests on one end on a stiff plank, one end of which can be raised; two edges of this end are parallel to the long edges of the plank, and provision is made to prevent slipping. What angle will the plank make with the horizontal when the block is on the point of overturning?

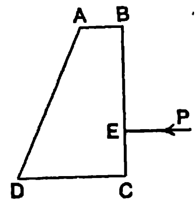

FIG. 142.

13. ABCD is the cross-section of a wall 40 feet long (Fig. 142). AB = 4 feet and is horizontal; BC = 15 feet and is vertical; CD = 9 feet and is horizontal. Find the centre of gravity of the wall. If the masonry weighs 150 lb. per cubic foot, find what horizontal force P, applied at a height CE = 5 feet above C, will just overturn the wall.

14. A solid uniform hemisphere rests with its curved surface in contact with a horizontal table. Show that the equilibrium is stable.

15. In Fig. 143, A is a semicylindrical body resting on a horizontal table. The top face of A is rectangular, 10 inches long in the direction perpendicular to the paper, and 4 inches in the direction parallel to the plane of the paper. B is a cylindrical rod made of the same kind of material as A, 2 inches diameter, and fixed perpendicularly to the centre of the top face of A. Find the height of B so that the equilibrium of the whole shall be neutral. (The centre of gravity of A is at a distance $4r/3\pi$ below the top face.)

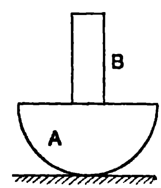

FIG. 143.

16. Draw an isosceles triangle, sides AB and AC 4 inches long, base BC 3 inches long. Bodies weighing 4, 6 and 8 lb. are fastened at A, B and C respectively. The triangle is made of a thin sheet weighing 1 lb. If the arrangement is suspended by a cord attached to the centre of AB, find and show in the drawing the position it will assume.

17. Find graphically the centre of gravity of the sheet shown in Fig. 144. AB is a chord drawn at a distance of 1 inch from the centre of the circular portion, and the radius of the circular portion is 3 inches.

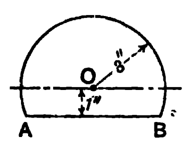

FIG. 144.

18. A body is placed first in one pan and then in the other pan of a false balance. When in the first pan, it is balanced by weights amounting to 0·562 lb. placed in the other pan; in the second operation, the weights amount to 0·557 lb. What is the true weight of the body? Assume that the balance beam swings correctly when both pans are empty. What is the error made by taking the arithmetical mean of the readings as the true weight?

19. A uniform lever weighing 85 grams rests on a knife edge at a point 7·3 cm. from its centre, and carries upon its longer end a weight of 105 grams, distant 23·3 cm. from the support, and a weight of 113 grams

18·4 cm. from the support. What weight must be carried on the shorter end at a point 21·7 cm. from the support in order that the lever shall be in equilibrium?

**20.** Prove that if a passenger of weight W advances a distance $a$ along the top of a motor-bus, a weight $Wa/b$ is transferred from the back springs to the front springs, where $b$ is the distance between the axles.

**21.** A uniform bar AB, 18 inches long, has a string AC, 7·5 inches long, attached at A, and another string BC, 19·5 inches long, attached at B. Both strings are attached to a peg C, and the rod hangs freely. Find graphically the angle which the rod makes with the horizontal.

**22.** The centre of gravity of a uniform semicircular sheet is at a distance of $4r/3\pi$ from the diametrical edge, $r$ being the radius of the semicircle. Deduce from this information the position of the centre of gravity of a uniform sheet in the shape of a quadrant of a circle. Explain clearly the method of deduction. L.U.

**23.** ABC is a horizontal lever pivoted at its middle point B, and carrying a scale-pan of weight $W_0$ at C; AD is a light bar pivoted at A to the lever and at D, vertically above A, to a horizontal bar FDE, which is freely movable about its end F, which is fixed. The weight of this bar is $W_1$, and its centre of gravity is at a distance $d$ from F and $FD = c$. Show how to graduate this bar with a movable weight $w$ for varying weights W placed in the scale-pan at C. If inch-graduations correspond to lb. wts. and $w = \frac{1}{4}$ lb., find the value of $c$. In this case find the relation between $W_0$ and $W_1$, when $d = 1$ inch and the zero mark is 1 inch from F. L.U.

**24.** Explain the meaning of the centre of a system of parallel forces, and show how to find it.

Weights of 1, 3, 4, 10 lb. respectively are placed at the corners of a square. Find the distance of their centre of gravity from each side of the square.

**25.** The axial distance between the wheels of a vehicle is 5 feet. The vehicle is loaded symmetrically, and the centre of gravity is at a height of 6 feet above the ground. Find the maximum angle with the vertical to which the vehicle may be tilted sideways without upsetting.

**26.** To determine the height of the centre of gravity of a locomotive, it is placed on rails, one of which is 5 inches above the other; and it is then found that the vertical forces on the upper and lower rails are respectively 23 and 37 tons. Calculate the height of the centre of gravity if the distance between the rails is 5 feet.

**27.** Prove that the sensitiveness of a balance is proportional to the length of the arm of the beam, and inversely proportional to the weight of the beam, and also inversely proportional to the distance between the centre of gravity of the beam and the central knife edge.

**28.** Show that the sensitiveness of a balance is independent of the load carried by the scale-pans when the terminal knife edges are in the same plane as the central knife edge. How does the behaviour of the balance depend upon the position of the centre of gravity of the beam?

C.W.B., H.C.

**29.** A regular pyramid whose base is a regular hexagon is made of uniform thin sheet metal. If $a$ is the radius of the circumcircle of its base, and $2a$ is the length of the slant edges, find the position of the centre of gravity of the total surface of the pyramid, and state its distance from the base.

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**30.** The distance between the scale-pan knife edges in a balance is 40 cm. The central knife edge is at a perpendicular distance of 1 cm. above the middle point of the line joining the scale-pan knife edges, and the centre of gravity of the moving part of the balance is 1 cm. below the middle point of the same line. Find the deflection of the beam when weights of 10·0 and 10·1 grams are placed in the scale-pans. The weight of the moving system is 1050 grams including the scale-pans weighing 25 grams each. Would the deflection still be the same if the weights in the scale-pans were 1·0 and 1·1 grams? Give reasons. J.M.B., H.S.C.

# CHAPTER X

## COUPLES. SYSTEMS OF UNIPLANAR FORCES

**Moment of a couple.**—In Fig. 145, $P_1$ and $P_2$ are equal parallel forces of opposite sense and therefore form a couple (p. 88). By taking moments successively about points A, B, C and D, it may be shown that **the couple has the same moment about any point in its plane.** Thus :

Taking moments about A :

Moment of the couple

$$=(P_1 \times 0) - (P_2 \times d) = -P_2 d, \quad \ldots(1)$$

the negative sign indicating an anticlockwise moment.

Taking moments about B :

Moment of the couple

$$=(P_2 \times 0) - (P_1 \times d) = -P_1 d. \quad \ldots(2)$$

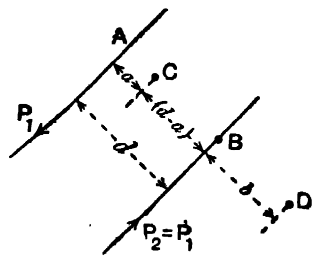



FIG. 145.—A couple has the same moment about any point in its plane.

Taking moments about C :

$$\text{Moment of the couple} = -(P_1 \times a) - P_2(d-a) = -P_2 d. \quad \ldots\ldots..(3)$$

Taking moments about D :

$$\text{Moment of the couple} = (P_2 \times b) - P_1(d+b) = -P_1 d. \quad \ldots\ldots\ldots..(4)$$

As $P_1$ and $P_2$ are equal, these four results are identical, thus proving the proposition. The perpendicular distance $d$ between the forces is called the **arm** of the couple.

**Equilibrant of a couple.**—**A couple may be balanced by another couple of equal and opposite moment applied $(a)$ in the same plane, or $(b)$ in a parallel plane.**

$(a)$ Reference is made to Fig. 146, in which are shown a clockwise couple, having equal forces $P_1$ and $P_2$ and an arm $a$, and an anticlockwise couple, having equal forces $Q_1$ and $Q_2$ and an arm $b$. Both couples are applied in the plane of the paper, and it is given that the moments $P_1 a$ and $Q_1 b$ are equal, or

$$Q_1 : P_1 = a : b. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Produce the lines of the four forces to intersect at A, B, C and D. From A draw AM and AN at right angles to $P_1$ and $Q_1$ respectively. Then AM $=a$ and AN $=b$. The triangles AMC and AND are similar, hence

$$AC : AD = AM : AN = a : b. \quad \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

*Therefore, from (1) and (2),*

$$Q_1 : P_1 = AC : AD. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

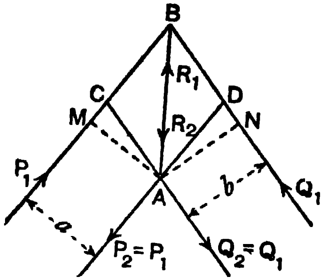



FIG. 146.—Equal opposing couples in the same plane are in equilibrium.

Now AC and BD are equal, and AD and BC are also equal, being opposite sides of a parallelogram; hence we may represent $P_1$ by CB, $P_2$ by DA, $Q_1$ by DB, and $Q_2$ by CA. Let $P_1$ and $Q_1$ act at B, then their resultant $R_1$ is represented by the diagonal AB. Let $P_2$ and $Q_2$ act at A, and their resultant $R_2$ is represented by BA. As $R_1$ and $R_2$ are equal, opposite and in the same straight line, they balance, and therefore the given couples balance.

(*b*) In Fig. 147 is shown a rectangular block having equal forces $P_1$ and $P_2$ applied to AD and BC respectively, and other equal forces $P_1$ and $P_2$ applied to the back edges FG and EH. These forces being all equal, the block is subjected to two equal opposing couples in parallel planes.

The resultant $R_1$ of the forces $P_1P_1$ will act perpendicularly to the bottom face and will bisect the diagonal DG; similarly, the resultant $R_2$ of $P_2P_2$ will bisect the diagonal BE and will be perpendicular to the top face. It is evident that $R_1$ and $R_2$ are equal and opposite, and that they act in the same straight line; hence they balance, and therefore the given couples balance.

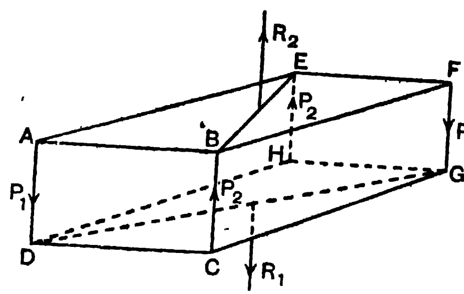



FIG. 147.—Equal opposing couples in parallel planes are in equilibrium.

**The effect of a couple is unaltered by shifting it to another position in the same plane or in a parallel plane.**—A couple may be balanced by the application in the same plane, or in a parallel plane, of a second couple of equal opposing moment; it follows that if the second couple be reversed its effect on the body will be identical with that of the first couple. Thus the second couple, so reversed, may be substituted for the first couple; in other words, the first couple may be shifted to any new position in the same plane, or in a parallel plane, without changing its effect on the body as a whole.

Further, the second couple need not have its forces equal to those

of the first couple, a fact which enables us to state that the forces of a given couple may be altered in magnitude, provided that the arm is altered to correspond, so as to leave the couple of unaltered moment. Thus, in Fig. 147, if the couple acting on the end EFGH has its forces given unequal to those of the couple acting on the end ABCD, equality of the forces may be obtained by making the arms of the couples equal.

The law that to every action there is always an equal and contrary reaction may now be extended by asserting that **to every couple there must be an equal and contrary couple, acting in the same or in a parallel plane.**

**Composition of couples in the same plane or in parallel planes.**—Any number of couples applied to a body and acting in the same plane, or in parallel planes, may be compounded by the substitution of a single couple having a moment equal to the algebraic sum of the moments of the given couples. This resultant couple may act in the given plane, or in any plane parallel to the given plane, without thereby altering the effect on the body as a whole.

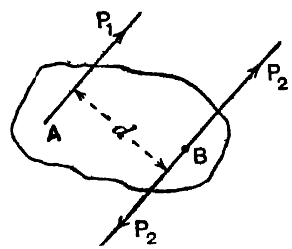


FIG. 148.—Transference of a force to a line parallel to the given line of action.

**Substitution of a force and a couple for a given force.**—In Fig. 148 is shown a body having a force $P_1$ applied at A. Suppose that it would be more convenient if $P_1$ were applied at another point B. To effect this change of position, let two opposing forces $P_2$, $P_2$, each equal to $P_1$, be applied at B in a line parallel to $P_1$; the forces $P_2$, $P_2$ will balance one another and therefore do not affect the given condition of the body. Let $d$ be the perpendicular distance between the lines of $P_1$ and $P_2$. The downward force $P_2$, together with $P_1$, forms a couple having a moment $P_1d$; this couple may be applied at any position in the same plane, leaving a force $P_2$ acting at B, equal to and having the same sense and direction as the given force $P_1$. A given force is therefore equivalent to an equal parallel force of the same sense together with a couple having a moment equal to the product of the given force and the perpendicular distance between the lines of the parallel forces.

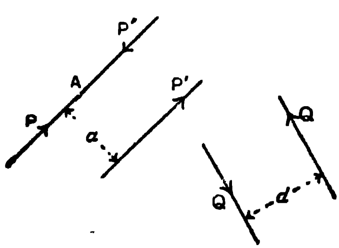


FIG. 149.—Reduction of a given force and couple to a single force.

**Substitution of a force for a given force and a given couple.**—In Fig. 149, a force P is given acting at A, also a couple having forces Q, Q

and an arm $d$. The system may be reduced to a single force by first altering the forces of the couple so that each is equal to P, the moment being kept unaltered by making $Qd = Pa$, where $a$ is the new arm of the couple. Let P′, P′ be the new forces of the couple, and apply the couple so that one of its forces acts in the same line as the given force P, and in the sense opposite to that of P. Then P and P′ acting at A balance each other, leaving a force P′ of the same sense as P, and acting in a line parallel to P and at a perpendicular distance $a$ from the line of P.

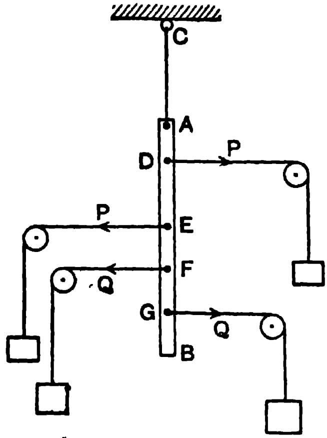



Fig. 150.—An experiment on couples.

Expt. 23.—**Equilibrium of two equal opposing couples.** In Fig. 150 is shown a rod AB hung vertically by a string attached at A and also to a fixed support at C. By means of cords, pulleys and weights, apply two equal, opposite and parallel forces P, P, and also another pair Q, Q; all these forces are horizontal. Adjust the values so that the following equation is satisfied:

$$P \times DE = Q \times FG.$$

Note that the rod remains at rest under the action of these forces.

Repeat the experiment, inclining the parallel forces P, P at any angle to the horizontal, and inclining the parallel forces Q, Q, to a different angle, but arranging that the moment of the P, P couple is equal to that of the Q, Q couple. Note whether the rod is balanced under the action of these couples.

Apply the P, P couple only, and ascertain by actual trial whether it is possible to balance the rod in its vertical position as shown in Fig. 150 by application of a single push exerted by a finger.

**Reduction of any system of uniplanar forces.**—In Fig. 151 are given four typical forces $P_1$, $P_2$, $P_3$ and $P_4$, acting in the plane of the paper at A, B, C and D respectively. Take any two rectangular axes OX and OY in the plane of the paper, and let the direction

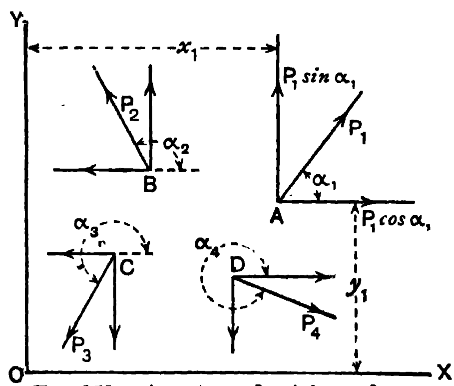



Fig. 151.—A system of uniplanar forces.

angles of the forces, $\alpha_1$, $\alpha_2$, etc., be stated with reference to OX. Resolve each force into components parallel to OX and OY; thus $P_1$ will have components $P_1 \cos \alpha_1$ and $P \sin \alpha_1$. Transfer into OX each component which is parallel to OX, and also transfer into OY each component which is parallel to OY. This will introduce a couple for each component so shifted; thus, the couple produced by shifting $P_1 \cos \alpha_1$ will be

$$(P_1 \cos \alpha_1) y_1$$

and that produced by shifting $P_1 \sin \alpha_1$ will be $(P_1 \sin \alpha_1) x_1$. Some of these couples will be clockwise and others anticlockwise; to obtain the resultant moment take the algebraic sum of the set parallel to OX and also the algebraic sum of those parallel to OY, giving:

Resultant moment of couples parallel to OX $= \Sigma(P \cos \alpha) y$.
Resultant moment of couples parallel to OY $= \Sigma(P \sin \alpha) x$.

The reduction of the system so far as we have proceeded is given in Fig. 152, and shows that we now have a number of forces in OX, another set in OY, and two couples.

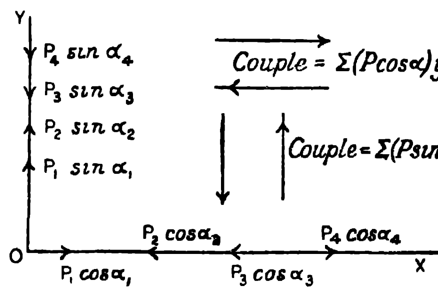


FIG. 152.—A system equivalent to that in Fig. 151.

Take the resultant $R_X$ of the forces in OX, also the resultant $R_Y$ of the forces in OY, giving:

$$R_X = \Sigma P \cos \alpha. \quad \text{.......(1)}$$
$$R_Y = \Sigma P \sin \alpha. \quad \text{.......(2)}$$

The resultant R of these forces will be given by

$$R = \sqrt{R_X^2 + R_Y^2}. \quad \text{.......(3)}$$

The angle $\alpha$ which R makes with OX (Fig. 153) will be determined from

$$\tan \alpha = \frac{R_Y}{R_X}. \quad \text{.......(4)}$$

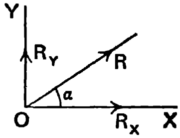


FIG. 153.

The system is now reduced to a force R and two couples. The resultant moment L of the two couples may be obtained by adding the couples algebraically; thus

$$L = \Sigma(P \cos \alpha) y + \Sigma(P \sin \alpha) x. \quad \text{.......(5)}$$

Let each force of this resultant couple be made equal to R, and let the arm be $a$; then

$$Ra = L, \quad \text{or} \quad a = \frac{L}{R}. \quad \text{.......(6)}$$

Apply the couple so that one of its forces R′ is in the same straight line as R acting at O (Fig. 154), and opposes R; the other force will be at a perpendicular distance OM $=a$ from O. It is evident that the two forces R, R′ at O balance; hence the given system has been reduced to a force R.

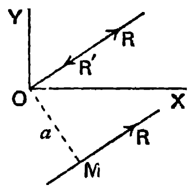



FIG. 154.– Resultant of the system.

**Special cases.**—Some special solutions of equations (1), (2) and (5) above may be examined. Suppose the result given by (5) to be zero; then the system reduces to a force acting at O. Should (2) also give zero, the system reduces to a force acting in OX, or to a force acting in OY if (1) be zero.

Should (5) have a numerical result and both (1) and (2) give zero, then the system reduces to a couple.

For **equilibrium**, there must be neither a resultant force nor a resultant couple; hence all three equations must give zero. The conditions of equilibrium may be written:

$$\Sigma \mathrm{P} \cos \alpha = 0. \quad \ldots\ldots\ldots\ldots (1)$$

$$\Sigma \mathrm{P} \sin \alpha = 0. \quad \ldots\ldots\ldots\ldots (2)$$

$$\Sigma (\mathrm{P} \cos \alpha) y + \Sigma (\mathrm{P} \sin \alpha) x = 0. \quad \ldots\ldots\ldots\ldots (3)$$

These equations must be satisfied simultaneously, and will serve for testing the equilibrium of any system of uniplanar forces.

The student will observe that equations (1) and (2) express the condition that a body in equilibrium does not suffer any displacement in consequence of the application of the force system, as would be the case if either or both the forces $\mathrm{R}_X$, $\mathrm{R}_Y$ had a magnitude other than zero. Equation (3) expresses the condition that no rotation of the body takes place as a consequence of the application of the forces. It will also be noted that equation (3) may be interpreted as the algebraic sum of the moments of the components of the given forces taken about an arbitrary point O.

The following typical examples should be studied thoroughly. In considering the equilibrium of a body, or of part of a body, care must be taken to show in the sketch those forces only which are applied to the body by agencies external to the body, and not those forces which the body itself exerts on other bodies.

EXAMPLE 1. AB and BC are smooth planes inclined respectively at 45° and 30° to the horizontal (Fig. 155 ($a$)). DE is a uniform rod 3 feet long and weighing 4 lb., and is maintained in a horizontal position by means of a body F, which has a weight of 2 lb. Where must F be placed?

Since the planes are smooth, the reactions, P and Q, of the planes are

perpendicular respectively to AB and BC. Resolve each force into horizontal and vertical components. Thus

$$P_x = P \sin 45° = \frac{P}{\sqrt{2}}; \quad P_y = P \cos 45° = \frac{P}{\sqrt{2}}.$$

$$Q_x = Q \sin 30° = \frac{Q}{2}; \quad Q_y = Q \cos 30° = \frac{Q\sqrt{3}}{2}.$$

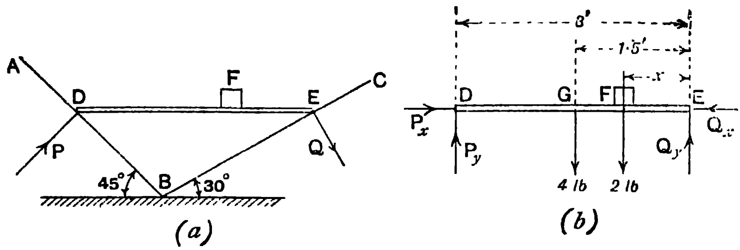



FIG. 155.—A rod resting on two inclined planes.

The rod is now acted upon by forces as shown in Fig. 155 (*b*). For equilibrium, we have

$$P_x - Q_x = 0; \qquad \therefore P_x = Q_x. \quad \ldots\ldots(1)$$

$$P_y + Q_y - 4 - 2 = 0; \qquad \therefore P_y + Q_y = 6. \quad \ldots\ldots(2)$$

Taking moments about E :

$$(P_y \times 3) - (4 \times 1\tfrac{1}{2}) - 2x = 0; \qquad \therefore 3P_y - 6 + 2x. \quad \ldots\ldots(3)$$

From (1),

$$\frac{P}{\sqrt{2}} = \frac{Q}{2}. \quad \ldots\ldots(4)$$

From (2),

$$\frac{P}{\sqrt{2}} + \frac{Q\sqrt{3}}{2} = 6. \quad \ldots\ldots(5)$$

Hence,

$$\frac{P}{\sqrt{2}} + \frac{P}{\sqrt{2}} \cdot \sqrt{3} = 6; \qquad \therefore P = \frac{6\sqrt{2}}{1 + \sqrt{3}};$$

$$\therefore P_y = \frac{P}{\sqrt{2}} = \frac{6}{1 + \sqrt{3}}. \quad \ldots\ldots(6)$$

Inserting this value in (3), we have

$$\frac{18}{1 + \sqrt{3}} = 6 + 2x,$$

whence $x = 0{\cdot}294$ foot.

EXAMPLE 2.—In Fig. 156 (*a*) AB and AC are two uniform bars each weighing 10 lb. and 6 feet long. The bars are smoothly jointed at A, and rest at B and C on a smooth horizontal surface. B and C are connected by an inextensible cord 4 feet long. A load of 40 lb.-weight is attached to D. Find, in terms of BD, the tension in the cord and the reactions communicated across the joint at A.

First consider ABC to be a rigid body, acted upon by vertical forces of 10, 40 and 10 lb.-weight, together with the reactions P and Q. Then

$$P + Q - 10 - 40 - 10 = 0 ; \quad \therefore \ P + Q = 60. \quad \ldots\ldots\ldots\ldots(1)$$

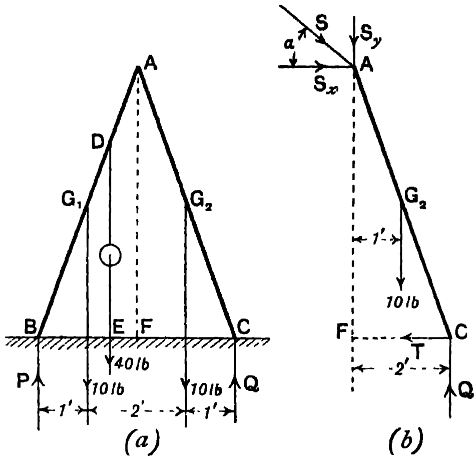



FIG. 156.—Equilibrium of two rods.

Taking moments about C gives

$$(P \times 4) - (10 \times 3) - (40 \times CE) - (10 \times 1) = 0 ;$$

$$\therefore \ 4P = 30 + 10 + 40 \,.\, CE;$$

$$P = 10 + 10CE. \quad \ldots\ldots\ldots\ldots(2)$$

From (1), $Q = 60 - P = 60 - 10 - 10CE$

$$= 50 - 10CE. \quad \ldots\ldots\ldots\ldots(3)$$

Also, $CE = BC - BE = 4 - BE.$

Draw AF perpendicular to BC, then, in the similar triangles BED, BFA we have

$$\frac{BE}{BD} = \frac{BF}{BA} = \frac{2}{6};$$

$$\therefore \ BE = \tfrac{1}{3}BD ;$$

$$\therefore \ CE = 4 - \tfrac{1}{3}BD.$$

Hence, from (2) and (3), $P = 10 + 10(4 - \tfrac{1}{3}BD)$

$$= 50 - \tfrac{10}{3}BD. \quad \ldots\ldots\ldots\ldots(4)$$

And $Q = 50 - 10(4 - \tfrac{1}{3}BD)$

$$= 10 + \tfrac{10}{3}BD. \quad \ldots\ldots\ldots\ldots(5)$$

Now consider the bar AC separately (Fig 156 (*b*)). The forces applied to it are its weight, acting vertically through $G_2$, the vertical reaction Q at C, the horizontal pull T of the cord BC, and a reaction S at A. S is exerted by the other bar AB, and its direction is guessed in Fig. 156 (*b*); the precise direction will be determined in the following calculation. Resolve S into

horizontal and vertical components $S_x$ and $S_y$, and apply the conditions of equilibrium.

$$T - S_x = 0 ; \quad \therefore \ T = S_x. \quad \text{.....(6)}$$

$$Q - 10 - S_y = 0 ; \quad \therefore \ Q = 10 + S_y. \quad \text{.....(7)}$$

Take moments about A, first calculating the length of AF :

$$AF = \sqrt{AC^2 - CF^2} = \sqrt{36 - 4} = \sqrt{32}.$$

$$(Q \times 2) - (10 \times 1) - (T \times \sqrt{32}) = 0 ;$$

$$\therefore \ 2Q = 10 + T\sqrt{32} \quad \text{.....(8)}$$

From (5) and (8), $\quad 2(10 + \frac{10}{3}BD) = 10 + T\sqrt{32}$ ;

$$\therefore \ T = \frac{10 + \frac{20}{3}BD}{\sqrt{32}},$$

$$= \frac{30 + 20BD}{3\sqrt{32}}. \quad \text{.....(9)}$$

From this result it will be seen that T increases if BD is made greater.

Again, from (6) and (9) :

$$S_x = T = \frac{30 + 20BD}{3\sqrt{32}}. \quad \text{.....(10)}$$

And from (5) and (7) :

$$10 + \frac{10}{3}BD = 10 + S_y ;$$

$$\therefore \ S_y = \frac{10}{3}BD. \quad \text{.....(11)}$$

(It will be observed from the positive sign of this result that the assumed direction of S has been chosen correctly.)

From (10) and (11) :

$$S = \sqrt{S_x^2 + S_y^2}$$

$$= \sqrt{\frac{3600BD^2 + 1200BD + 900}{288}}.$$

Also, the angle $\alpha$ which S makes with the horizontal is given by

$$\tan \alpha = \frac{S_y}{S_x} = \frac{\frac{10}{3}BD}{\frac{30 + 20BD}{3\sqrt{32}}} = \frac{10 \, . \, BD\sqrt{32}}{30 + 20BD}.$$

EXAMPLE 3.—In Fig. 157, AB is a light rod having the end A guided so as to be capable of moving freely in a horizontal line AD. At C another light bar CD is smoothly jointed to AB ; CD can turn freely about D. A load W is hung from B. If AC = CD and BC = $n$ . AC, find the horizontal and vertical reactions which must be applied at A in order to maintain the arrangement with AB at an angle $\theta$ to the horizontal.

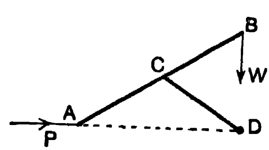



FIG. 157.

Consider the equilibrium of the rod AB. Let Q be the reaction which DC applies to C. Take horizontal and vertical components of Q and let these be $Q_x$ and $Q_y$ (Fig. 158) Since $Q_x$ and AD

are parallel, the angle between Q and $Q_x$ is equal to the angle ADC; also the angles ADC and CAD are equal, since AC = CD. Hence the angle between $Q_x$ and Q = $\theta$. Therefore

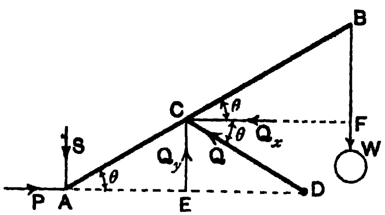



FIG. 158.—Forces acting on AB.

$$Q_x = Q\cos\theta, \quad \text{and} \quad Q_y = Q\sin\theta.$$

For equilibrium,

$$P - Q_x = 0;$$

$$\therefore\ P = Q_x. \quad \ldots\ldots\ldots\ldots(1)$$

$$Q_y - W - S = 0;$$

$$\therefore\ Q_y = W + S. \quad \ldots\ldots\ldots(2)$$

Taking moments about C:

$$(P \times CE) + (S \times AE) - (W \times CF) = 0;$$

$$\therefore\ (P \times CE) + (S \times AE) = W \times CF. \quad \ldots\ldots\ldots\ldots(3)$$

From (3): $\quad P \,.\, AC\sin\theta + S \,.\, AC\cos\theta = W \,.\, BC\cos\theta.$

Dividing this throughout by AC cos $\theta$, we obtain

$$P\tan\theta + S = W\frac{BC}{AC} = \frac{Wn \,.\, AC}{AC} = Wn. \quad \ldots\ldots\ldots\ldots(4)$$

From (2), $\quad Q\sin\theta = W + S.$

From (1), $\quad P = Q\cos\theta, \quad \text{or} \quad Q = \dfrac{P}{\cos\theta};$

$$\therefore\ \frac{P\sin\theta}{\cos\theta} = P\tan\theta = W + S. \quad \ldots\ldots\ldots\ldots(5)$$

Substituting in (4), we have

$$W + S + S = Wn;$$

$$\therefore\ 2S = Wn - W,$$

$$S = W\left(\frac{n-1}{2}\right). \quad \ldots\ldots\ldots\ldots(6)$$

Hence, from (5) and (6),

$$P\tan\theta = W + W\left(\frac{n-1}{2}\right),$$

$$P = \frac{W\left\{1 + \dfrac{n-1}{2}\right\}}{\tan\theta}$$

$$= \left(\frac{n+1}{2\tan\theta}\right)W. \quad \ldots\ldots\ldots\ldots(7)$$

It will be noted from (6) that if $n = 1$, *i.e.* AC, CD and CB are all equal (Fig. 157), then S = 0. Inspection of Fig. 158 shows that under these conditions the line of W passes through D; hence P, Q and W intersect in a point and can equilibrate the rod AB without the necessity for the application of a force S. If $n$ be less than 1, the result given for S by (6) is negative, and indicates that S must act upwards.

The graphical solution of this problem depends upon the fact that the rod AB (Fig. 158) is acted upon by three forces, viz. W, Q and the resultant

of P and S. The point at which these forces intersect may be found by producing W and Q; the resultant of P and S then passes through this point, and also through A. The solution is then obtained by application of the triangle of forces, and will be found to be an interesting problem.

## Exercises on Chapter X

1. A rectangular plate, 6 inches by 2 inches, has a force of 400 lb. weight applied along a long edge. Show how to balance the plate by means of forces acting along each of the other edges. Neglect the weight of the plate.

2. A door weighs 120 lb. and has its centre of gravity in a vertical line parallel to and at a distance of 18 inches from the axis of the hinges. The hinges are 4 feet apart and share the vertical reaction required to balance the door equally between them. Find the reaction of each hinge.

3. A vertical column has a bracket fixed to its side near the top. A load of 5 tons weight hangs from the end of the bracket at a point 8 inches from the axis of the column. Remove this load and apply an equivalent system of forces consisting partly of a force of 5 tons weight acting in the axis of the column. Show the system in a sketch.

4. Sketch a right-angled triangle in which AB = 16 feet and is vertical, and BC = 10 feet and is horizontal. The triangle represents the cross-section of a wall 10 feet long and weighing 140 lb. per cubic foot. Find the reaction of the foundation of the wall, expressed as a force acting at the centre of the base together with a couple.

5. A rod AB, 4 feet long, has a pull of 20 lb. weight applied at A in a direction making 30° with AB. There is also a couple having a moment of 40 lb.-ft. acting on the rod. Find the resultant force.

6. Draw any triangle ABC. Forces act in order round the sides of the triangle, and each force has a magnitude proportional to the length of the side along which it acts. Reduce the system of forces to its simplest form.

7. A square plate ABCD of 2 feet edge has forces acting along the edges as follows: From A to B, 2 lb. weight; from B to C, 3 lb. weight; from C to D, 4 lb. weight; from D to A, 5 lb. weight. Find the resultant.

8. A uniform rod AB is 4 feet long and weighs 24 lb. The end A is smoothly jointed to a fixed support; the rod is inclined at 45° and its upper end B rests against a smooth vertical wall. A load of 10 lb. weight is hung from a point in the rod 1 foot from A. Find the reactions at A and B.

9. In an isosceles triangle AC = CB; AB is 15 feet and is horizontal; C is 5 feet vertically above AB. The plane of the triangle is vertical, and the triangle is supported at A and B. A load of 400 lb. weight is applied at the centre of AC, another of 600 lb. weight at C, and a force of 800 lb. weight acts at the centre of BC at 90° to BC. The reaction of the support at B is vertical; that at A is inclined. Find the reactions of both supports.

10. In the arrangement shown in Fig. 157 (p. 121), AC = CD = 4 inches, BC = 6 inches. Find P, S and Q when a load of 10 lb. weight is hung from B for values of $\theta$ of 45°, 30°, 15°, 5°.

11. Answer Question 10, (*a*) if BC = 4 inches; (*b*) if BC = 3 inches.

12. BC is a rod 12 inches long and capable of turning in the plane of the paper about a smooth pin at C. Another rod AB, 4 feet long, is jointed smoothly to BC at B; the end A can travel in a smooth groove, which,

when produced, passes through C. The angle ACB is 30°, and a load of 200 lb. weight is hung from the centre of AB. Calculate the resultant force which must be applied at A in order to preserve the equilibrium. Check the result by solving the problem graphically.

**13.** A ladder, 20 feet long, is inclined at 60° to the horizontal, and rests on the ground at A and against a wall at B. The ladder weighs 80 lb., and its centre of gravity is 8 feet from A. Assuming both ground and wall to be smooth, the reactions at A and B will be vertical and horizontal respectively. The ladder is prevented from slipping by means of a rope attached at A and to a point at the foot of the wall. A man weighing 150 lb. ascends the ladder. Calculate the pull in the rope when he is 4, 8, 12, 16 and 19 feet from A. Plot a graph showing the relation of the pull and his distance from A.

**14.** Show how to find the resultant of two unequal parallel forces acting at different points but in opposite directions upon a rigid body. Is there a single resultant if the two forces are equal?

A steel cylindrical bar weighing 1000 lb. is held in a vertical position by means of two thin fixed horizontal planks 5 ft. apart vertically, in which are holes through which the bar can slide. If the sides of these holes are smooth and the bar is lifted by a vertical force applied 2 inches from its axis, find the pressure on each plank.

**15.** If a number of uniplanar forces act upon a rigid body, prove that they will be in equilibrium, provided that the algebraic sum of their resolved parts in two directions at right angles and of their moments about one given point in the plane be zero.

ABCD is a square lamina. A force of 2 lb. weight acts along AB, 4 lb. weight along BC, 6 lb. weight along CD, 8 lb. weight along DA, $2\sqrt{2}$ lb. weight along CA, and $4\sqrt{2}$ lb. weight through the point D parallel to AC. Find the resultant of the system of forces.

**16.** A uniform plank, 12 feet long, weighing 40 lb., hangs horizontally, and is supported by two ropes sloping outwards; the ropes make angles of 60° and 45° respectively with the horizontal. If the plank carries a weight of 100 lb., find where this weight must be placed.

**17.** Four equal uniform rods, each of weight W, are jointed so as to form a square ABCD. The arrangement is hung from a cord attached at A, and the corners B and D are connected by a light rod so that the square retains its shape. Show that the thrust along the rod BD is 2W, and find the reaction at the bottom hinge.

**18.** Define the expression 'moment of a force about a point'. Show from your definition that the sum of the moments of two equal and parallel forces acting in opposite directions is the same about every point in the plane in which they act.

**19.** Each half of a step-ladder is $5\frac{1}{2}$ feet long, and the two parts are connected by a cord 28 in. long, attached to points in them distant 16 in. from their free extremities. The half with the steps weighs 16 lb., and the other half weighs 4 lb. Find the tension in the cord when a man weighing 11 stone is standing on the ladder, $1\frac{1}{2}$ ft. from the top, it being assumed that the cord is fully stretched, and that the reactions between the ladder and the ground are vertical. L.U.

**20.** A bar AB of weight W and length $2a$ is connected by a smooth hinge at A with a vertical wall, to a point C of which, vertically above A,

and such that AC = AB, B is connected by an inextensible string of length $2l$. Find in terms of these quantities the tension of the string and the action at the hinge.

If the bar is 6 ft. long and weighs 10 lb., and the string be 2 ft. long, show that its tension is $1\frac{2}{3}$ lb. wt. And if the joint at A be slightly stiff, so that in addition to the supporting force it exerts a couple G reducing the tension in the string to 1 lb. wt., then G will be approximately 3·94 lb.-ft. units. L.U.

**21.** ABC and ACD are two triangles in which AB = AC = AD, and the angles BAC and CAD are 60° and 30° respectively. With AD vertical, the figure represents a wall bracket of five light rods. The bracket is fixed at A, and kept just away from the wall by a small peg at D, and a mass of 5 pounds is suspended from B. Find, preferably by analytical methods, the pressure on the peg at D and the forces in the rods DC, CA and AB, stating whether they are in compression or tension. L.U.

**22.** Two ladders, AB and AC, each of length $2a$, are hinged at A and stand on a smooth horizontal plane. They are prevented from slipping by means of a rope of length $a$ connecting their middle points. If the weights of the ladders are 40 and 10 lb., find the tension in the rope, and the horizontal and vertical components of the action at the hinge.

**23.** A rod AB can turn freely about A, and is smoothly jointed at B to a second rod BC, whose other end C is constrained to remain in a smooth groove passing through A. A force F is applied to BC along CA. Prove that the couple produced on AB is F × AK, where K is the point in which BC produced cuts the perpendicular at A to AC.

**24.** Define a couple. What is the characteristic property of couples? Prove that a couple and a force acting in the same plane are equivalent to a single force. Three forces act along the sides of a triangular lamina and are proportional to the sides along which they act; find the magnitude of their resultant.

**25.** Six equal light rods are jointed together at their ends so as to form a regular hexagon ABCDEF. C and F are connected together by another light rod. The arrangement is hung by two vertical cords attached to A and B respectively so that AB, CF and DE are horizontal. Equal weights of 20 lb. each are hung from D and E. Find the forces in each rod, and state whether they are pulls or pushes.

**26.** Prove that the moment of two non-parallel coplanar forces about any point in the plane is equal to the moment of their resultant. Prove also that, if forces act along the sides of a plane polygon taken in order, each force being represented in magnitude and direction by the side along which it acts, they are equivalent to a torque (*i.e.* turning moment) represented by twice the area of the polygon.

**27.** Chains of length 3 ft. and 4 ft. are attached to the ends of a uniform beam of length 5 ft. and mass 200 lb. The other ends of the chains are attached to a crane and the whole hangs in mid-air in equilibrium. Find the inclination of the beam to the horizontal and the tensions in the two chains. (The masses of the chains can be neglected.) J.M.B., H.S.C.

**28.** Two equal uniform rods AB and AC, each of length 4 feet and weight 5 lb., are smoothly jointed at A. They are placed symmetrically on a smooth fixed sphere of radius 1 foot, so that A is vertically above the centre. Show that the angle between the rods when in equilibrium is a right angle, and find the magnitude of the reaction at the joint.

J.M.B., H.S.C.

**29.** A, B, C (Fig. 159) are three toothed wheels which revolve in contact at their edges, A and C being free to turn about their common centre, and the teeth of C being turned inwards from a projecting rim so as to catch on the teeth of B. Couples $K_a$, $K_b$, $K_c$ are applied to them respectively in the same sense round their axes and when the wheel B is free to move round in contact with the wheels A and C; prove that the wheels will be in equilibrium if

$$K_a/a = K_b/2b = -K_c/c.$$

J.M.B., H.S.C.

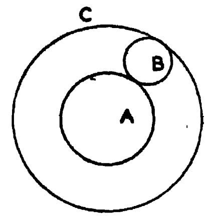



FIG. 159.

# CHAPTER XI

## STRESS. STRAIN. ELASTICITY

**Stress.**—The term **stress** is applied to the mutual actions which take place across any section of a body to which a system of forces is applied. Stresses are described as **tensile** or **pull**, **compressive** or **push**, and **shear**, according as the portions of the body tend to separate, to come closer together, or to slide on one another respectively.

If equal areas at every part of the section sustain equal forces, the stress is said to be **uniform**; otherwise the stress is **varying**. Stress is measured by the force per unit area, and is calculated by dividing the total force by the area over which it is distributed; the result of this calculation is called the **stress intensity**, or more usually simply the **stress**. In the case of varying stress, the result of the above calculation gives the average stress intensity. In such cases the stress at any point is calculated by taking a very small area embracing the point and dividing the force acting over this area by the area.

Common units of stress are the pound-, or ton-weight per square inch, or per square foot. In the C.G.S. system the dyne per square centimetre is the unit of stress; the kilogram weight per square centimetre is the practical metric unit, and is equivalent to 14·19 lb. weight per square inch. The dimensions of stress are

$$\frac{ml}{t^2} \div l^2 = \frac{m}{lt^2} \quad \text{or} \quad ml^{-1}t^{-2}.$$

**Strain.**—The term **strain** is applied to any change occurring in the dimensions, or shape of a body when forces are applied. A rod becomes longer or shorter during the application of pull or push, and is said to have **longitudinal strain**. This strain is calculated as follows:

Let $L$ = the original length of the rod,
$e$ = the alteration in length, both expressed in the same units.

Then, $$\text{Longitudinal strain} = \frac{e}{L}.$$

A body subjected to uniform normal stress (hydrostatic stress) all over its surface has **volumetric** strain.

Let $V$ = the original volume of the body;
$v$ = the change in volume, both expressed in the same units.

Then, $$\text{Volumetric strain} = \frac{v}{V}.$$

**Shearing strain** occurs when a body is subjected to shear stress. In this kind of stress a change of shape occurs in the body. Thus, hold one cover of a thick book firmly on the table, and apply a shearing force to the top cover (Fig. 160). The change in shape is rendered

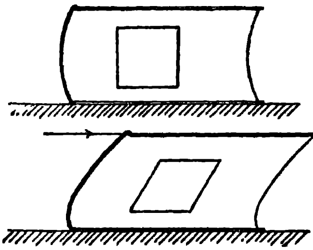


FIG. 160.—An illustration of shearing strain.

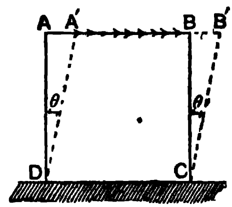



FIG. 161.—Shearing strain.

evident by the square, pencilled on the end of the book, becoming a rhombus. Under similar conditions, a solid body would behave in the same manner, but in a lesser degree (Fig. 161). Shearing strain is measured by stating the angle $\theta$ in radians through which the vertical edge in Fig. 161 has rotated on application of the shearing stress. For metals $\theta$ is always very small, and it is sufficiently accurate to write

$$\text{Shearing strain} = \theta = \frac{BB'}{BC}.$$

It will be noted that strain has zero dimensions.

**Elasticity.**—**Elasticity** is that property of matter by virtue of which a body endeavours to return to its original shape and dimensions when strained, the recovery taking place when the disturbing forces are removed. The recovery is practically perfect in a great many kinds of material, provided that the body has not been loaded beyond a certain limit of stress which differs for different materials. If loaded beyond this **elastic limit** of stress, the recovery of the original shape and dimensions is incomplete, and the body is said to have acquired **permanent set.**

**Hooke's law.**—Experiments on the pulling and pushing of rods show that the change in length is proportional very nearly to the force applied. If one end of a rod is held firmly while the other end is twisted, it is found that the angle through which this end rotates relatively to the fixed end is proportional to the twisting moment applied. Experimental evidence shows that beams are deflected, and springs are extended, by amounts proportional to the loads applied. This law was discovered by Hooke and bears his name. Since in every case the stress is proportional to the load, and the strain is proportional to the

change in dimension, **Hooke's law** may be stated thus : **Strains are proportional to the stresses producing them.**

Hooke's law is obeyed by a great many materials up to a certain limit of stress, beyond which strains are produced which are larger proportionally than those for smaller stresses. In **ductile materials**, such as wrought iron, which are capable of being wire-drawn, rolled, and bent, the point of breakdown of Hooke's law marks the beginning of a **plastic state** which, when fully developed, is evidenced by a large strain taking place with practically no increase in the stress. The stress at which this large increase in strain occurs is called the **yield point**, and is considerably greater than the stress at which Hooke's law breaks down.

Experiments for the determination of the stress at which a given material first acquires permanent set are tedious, and when the term 'elastic limit' is used, it is generally understood to mean the stress at which Hooke's law breaks down; the latter stress is determined easily by experiment.

**Modulus of elasticity.**—Assuming that Hooke's law is obeyed by a given material, and that $s$ is the strain produced by a given stress $p$, we have

$$s \propto p, \quad \text{or} \quad as = p,$$

$$\therefore \; a = \frac{p}{s}, \quad \text{......(1)}$$

where $a$ is a constant for the material considered, and is called a **modulus of elasticity**. The value of the modulus of elasticity depends upon the kind of material and the nature of the stress applied. There are three chief moduli of elasticity.

**Young's modulus** applies to a pulled or pushed rod, and is obtained by dividing the stress on a cross-section at 90° to the length of the rod by the longitudinal strain.

Let $\mathsf{P}$ = the pull or push applied to the rod, in units of force.
$\mathsf{A}$ = the area of the cross-section.
$\mathsf{L}$ = the original length of the bar.
$e$ = the change in length of the bar.

Writing $\mathsf{E}$ for Young's modulus, we have

$$\mathsf{E} = \frac{\text{stress}}{\text{strain}} = \frac{\mathsf{P}}{\mathsf{A}} \div \frac{e}{\mathsf{L}} = \frac{\mathsf{PL}}{\mathsf{A}e}. \quad \text{......(2)}$$

The **bulk modulus** applies to the case of a body having uniform normal stress distributed over the whole of its surface.

Let $p$ = the stress intensity.
$\mathsf{V}$ = the original volume of the body.
$v$ = the change in volume.

Writing K for the bulk modulus, we have

$$K=\frac{\text{stress}}{\text{volumetric strain}}=p\div\frac{v}{V}=\frac{pV}{v}. \qquad\text{.....................(3)}$$

MODULI OF ELASTICITY

*(Average values)*

| Material | Young's modulus, E | | Rigidity modulus, $n$ | |
|---|---|---|---|---|
| | Dynes per sq. cm. | Tons per sq. inch | Dynes per sq. cm. | Tons per sq. inch |
| Cast iron - - - | $10 \times 10^{11}$ | 6,000 | $3{\cdot}5 \times 10^{11}$ | 2,200 |
| Wrought iron - - | 20 ,, | 13,000 | 8·1 ,, | 5,200 |
| Mild steel - - - | 20 ,, | 13,500 | 8·5 ,, | 5,500 |
| Copper (rolled) - - | 9·5 ,, | 6,200 | 3·9 ,, | 2,500 |
| Aluminium (rolled) - | 6·2 ,, | 4,000 | 2·5 ,, | 1,600 |
| Brass - - - - | 9·0 ,, | 5,700 | 3·4 ,, | 2,200 |
| Gun-metal - - - | 7·8 ,, | 5,000 | 3·1 ,, | 2,000 |
| Phosphor bronze - - | 9·3 ,, | 6,000 | 3·6 ,, | 2,300 |
| Timber - - - | 1·1 ,, | 700 | — | — |
| Indiarubber - - - | 0·05 ,, | 32 | 0·0002 ,, | 0·13 |
| Glass, Crown - - | 7·0 ,, | 4,500 | 3·0 ,, | 1,940 |
| ,, Flint - - - | 5·5 ,, | 3,500 | 2·2 ,, | 1,420 |
| Catgut - - - | 0·3 ,, | 194 | — | — |

The rigidity modulus applies to a body under shearing stress.

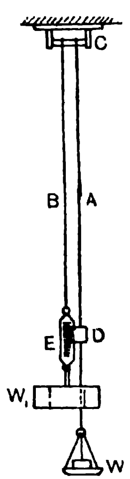



FIG. 162.—Apparatus for tensile tests on wires.

Let $q$ = the shearing stress intensity.
$\theta$ = the shear strain.

Writing $n$ for the rigidity modulus, we have

$$n=\frac{q}{\theta}.$$

The dimensions of all these moduli are the same as those of stress. The numerical values are expressed in the same units as those used in stating the stress (p. 127).

EXPT. 24.—**Elastic stretching of wires.** A simple type of apparatus is shown in Fig. 162. Two wires, A and B, are hung from the same support, which should be fixed to the wall as high as possible in order that long wires may be used. One wire, B, is permanent and carries a fixed load $W_1$ in order to keep it taut. The other wire, A, is that under test, and may be changed readily

for another of different material. The extension of A is measured by means of a vernier D, clamped to the test wire and moving over a scale E, which is clamped to the permanent wire. The arrangement of two wires prevents any drooping of the support being measured as an extension of the wire.

See that the wires are free from kinks. Measure the length L, from C to the vernier. Measure the diameter of the wire A in several places and take the mean value. State the material of the wire, and also whatever is known of its treatment before it came into your hands. Apply a series of gradually increasing loads to the wire A, and read the vernier after the application of each load. If it is not desired to reach the elastic limit, stop when a maximum safe load has been applied, and obtain confirmatory readings by removing the load step by step. In order to obtain the elastic limit, the load should be increased by small increments, and the test stopped when it becomes evident that the extensions are increasing more rapidly than the loads. Tabulate the readings thus:

TENSION TEST ON A WIRE

| Load, lb. or kilograms wt. | Vernier reading | | Extension, inches or mm. |
|---|---|---|---|
| | Load increasing | Load decreasing | |
| | | | |

Plot loads as ordinates, and the corresponding extensions as abscissae (Fig. 163). It will be found that a straight line will pass through most of the points between O and a point A, after which the graph turns towards the right. The point A indicates the breakdown of Hooke's law.

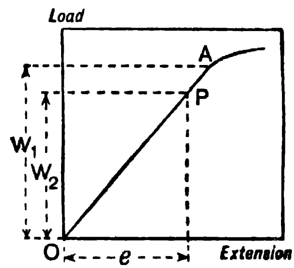


FIG. 163.—Graph of a tensile test on a wire.

Let $W_1$ = load at A in Fig. 163,

$d$ = the diameter of the wire.

Then, Stress at elastic breakdown $= W_1/\frac{1}{4}\pi d^2$.

Select a point P on the straight line OA (Fig. 163), and measure $W_2$ and $e$ from the graph.

Let $W_2$ = the load at P.

$e$ = the extension produced by P.

L = the length of the test wire.

Then,

$$\text{Young's modulus} = E = \frac{\text{stress}}{\text{strain}} = \frac{W_2}{\frac{1}{4}\pi d^2} \cdot \frac{L}{e}.$$

**Pure torsion.**—In Fig. 164 is shown a rod AB having arms CD and EF fixed to it at right angles to the length of AB. Let AC = AD = BE = BF, and let equal opposite parallel forces Q, Q be applied at C and D in directions making 90° with CD. Let other equal opposite parallel forces P, P be applied in a similar manner at E and F. The rod is then under the action of two opposing couples in parallel planes. If the forces are all equal, the couples have equal moments, and the system is in equilibrium (p. 113). The rod is then said to be under **pure torsion,** *i.e.* there is no tendency to bend it, and there is no push or pull in the direction of its length. The **twisting moment** or **torque** T is given by the moment of either couple, thus

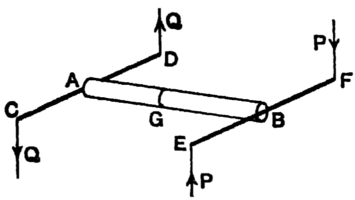



FIG. 164.—Pure torsion.

$$\text{Torque} = T = Q \times CD = P \times EF.$$

The actions of the couples are transmitted from end to end of the rod AB, and produce shearing stresses on any cross-section, such as G (Fig. 164).

The following experiment illustrates the twisting of a wire under pure torsion.

EXPT. 25.—**Torsion of a wire.** In Fig. 165, AB is a wire fixed firmly at A to a rigid clamp and carrying a heavy cylinder at B. The cylinder serves to keep the wire tight, and also provides means of applying a twisting couple to the wire. Two cords are wound round the cylinder and pass off in opposite parallel directions by guide pulleys. Equal weights $W_1$ and $W_2$ are attached to the ends of the cords. Pointers C and D are clamped to the wire and move, as the wire twists, over fixed graduated scales E and F. The angle of twist produced in the portion CD of the wire is thus indicated.

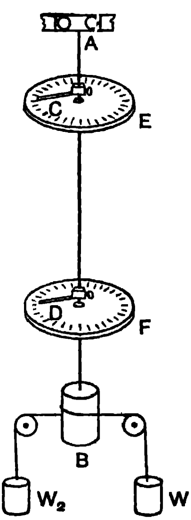



FIG. 165.—Apparatus for torsion tests on wires.

State the material of the wire; measure its diameter $d_1$ by means of a screw gauge at a number of places along its length and take the average value in the calculation. Also measure the length $l$ between the pointers C and D. Measure the diameter $d_2$ of the cylinder B. Apply a series of gradually increasing loads, and read the scales E and F after each load is applied. Tabulate the readings.

### Experiment on Torsion

| Load, $W_1 = W_2$ | Torque, $W_1 d_2$ | Scale readings, degrees | | Angle of twist, degrees |
|---|---|---|---|---|
| | | Scale E | Scale F | |
| | | | | |

Plot the torques as ordinates and the corresponding angles of twist as abscissae. A typical graph is shown in Fig. 166. A straight line graph indicates that the angle of twist $\theta$ is proportional to the torque. Select a point P on the graph, and scale the torque T and the angle $\theta$. If the graph has been plotted in degrees, convert $\theta$ to radians. Calculate the modulus of rigidity from the expression :

$$n = \frac{32Tl}{\pi d_1^4 \theta}.$$

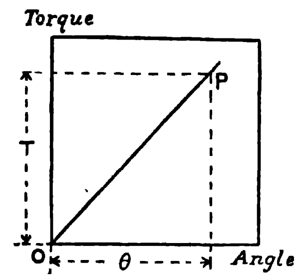



Fig. 166.—Graph of a torsion test on a wire.

Expt. 26.—**Stretching of a spring.** When a weight is suspended by a spring of thin wire wound into a spiral of small pitch, the strain is almost pure shear and the extension of the spring is proportional to the angle of shear. Suspend such a spring from a hook and fix a scale-pan to its lower end. Add weights to the pan, measuring the extension of the spring for each weight added. Show by plotting a graph that the extension is proportional to the load producing it.

**Couple due to torsion.**—In Fig. 167 a section of a circular wire or rod is seen, exaggerated in size. When torsion takes place, every cylindrical layer of the wire of radius $r$ and thickness $dr$ undergoes a shearing strain and a couple is required to maintain this strain in the layer. The sum of all the couples in the various layers of the wire constitutes the resultant couple required to produce the torsion.

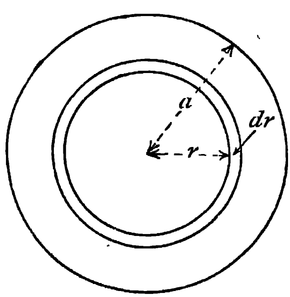



Fig. 167.—Section of wire.

In order to find this couple, consider a strip AB of the layer (Fig. 168), of length $l$ and width $b$. If the upper end of the wire is fixed and the lower end rotated through angle $\theta$, B moves through distance $r\theta$, and the angle of shear in the strip AB is $\frac{r\theta}{l}$. The area of cross-section

of the strip is $b \,.\, dr$ and if $f$ is the force applied over this area, parallel to its plane, the shearing stress is $\frac{f}{b \,.\, dr}$, and the modulus of rigidity $n$ is given by

$$n = \frac{\text{stress}}{\text{strain}} = \frac{f}{b \,.\, dr} \div \frac{r\theta}{l};$$

$$\therefore f = \frac{nr\theta b}{l} dr.$$

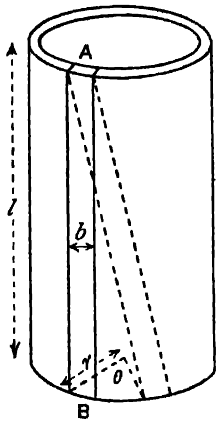



FIG. 168.—Strain in a wire undergoing torsion.

There is a force exactly equal to this but in the opposite direction, due to the strain in the similar strip diametrically opposite to AB, and the two constitute a couple $\frac{2nr^2\theta b}{l} dr$. Proceeding round half the circumference of the wire and adding up the corresponding couples for the whole layer gives a couple found by replacing $b$ by $\pi r$. Thus the couple for the whole layer is $\frac{2\pi n\theta}{l} r^3 dr$.

In order to add up all the couples for the layers from the axis of the wire to the surface, of radius $a$, a graph connecting $r$ and $r^3$ may be drawn, from $r=0$ to $r=a$, and the area between it and the axis of $r$ found. This gives the sum of all the values of $r^3 dr$, and, multiplying it by $\frac{2\pi n\theta}{l}$, the total couple is found. It is better, however, to perform a simple integration, thus:

$$\text{Total couple} = \frac{2\pi n\theta}{l} \int_0^a r^3 dr$$

$$= \frac{2\pi n\theta}{l} \left[\frac{r^4}{4}\right]_0^a$$

$$= \frac{\pi n a^4}{2l} \theta.$$

It is thus seen that the couple for a given wire is proportional to $\theta$.

**Bending of a beam.** The beam shown in Fig. 169 consists of a number of planks of equal lengths laid one on the top of the other and supported

FIG. 169.—Bending of a loose plank beam.

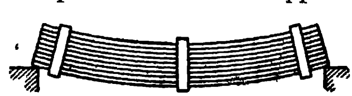


FIG. 170.—Bending of a strapped plank beam.

at the ends. Application of a load causes all the planks to bend in a similar fashion, and the planks will now be found to overlap at the ends.

Strapping the planks together (Fig. 170) prevents this action, and each end of the beam now lies in one plane ; the planks now behave approximately like a solid beam. Inspection of Fig. 170 shows that planks near the top have become shorter and those near the bottom have become longer. The middle plank does not change in length. Hence we may infer that, in solid beams, there is a **neutral layer** which remains of unaltered length when the beam is bent, and that layers above the neutral layer have longitudinal strain of shortening, and must therefore be under push stress. Layers below the neutral layer have longitudinal strain of extension and are therefore under pull stress.

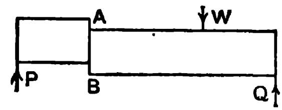


FIG. 171.—Shear at the section AB.

In Fig. 171 the beam carries a load W, and the reactions of the supports are P and Q. Considering any cross-section AB, the actions of P on the portion of the beam lying on the left-hand side of AB, and of W and Q on the other portion, produce a tendency for the material at AB to slide as shown. Hence the material at AB is under shear stress.

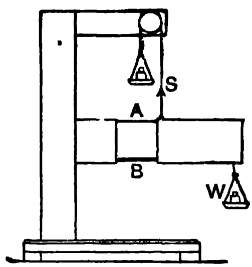


FIG. 172.—Model of a cantilever, cut to show the forces at AB.

Beams firmly fixed in and projecting from a wall or pier are called **cantilevers**. A model cantilever is shown in Fig. 172, and is arranged so as to give some conception of the stresses described above. The cantilever has been cut at AB ; in order to balance the portion outside AB, a cord is required at A (indicating pull stress) and a small prop at B (indicating push stress). Further, in order to balance the tendency to shear, a cord has been arranged so as to apply a force S. In the uncut cantilever, these forces are supplied by the stresses in the material.

**Bending moment and shearing force in beams.**—In Fig. 173 (*a*) is shown a beam carrying loads $W_1$, $W_2$, and supported by forces P, Q. AB is any cross-section. P and $W_1$ have a tendency to rotate the portion of the beam lying on the left-hand side of AB. Similarly, Q and $W_2$ tend to rotate the other portion of the beam in the contrary sense. These tendencies may be calculated by taking the algebraic sum of the moments of the forces about any point in AB. A little consideration will

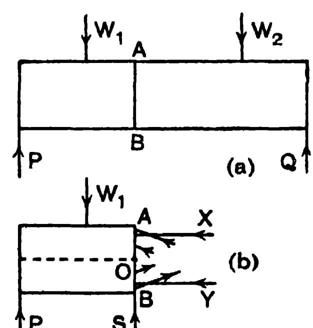


FIG. 173.—Bending moment and shearing force.

show that the resultant moment of P and $W_1$ must be equal to the resultant moment of Q and $W_2$, since both these resultant moments are balanced by the same stresses transmitted across AB. The evaluation of these stresses is beyond the scope of this book, but we may say that they give rise to equal forces X and Y (Fig. 173 (*b*)).

The **bending moment** at any section of a beam measures the tendency to bend the beam about that section, and is calculated by taking the algebraic sum of the moments about any point on the section of all the forces applied to either one portion or the other portion of the beam.

Again, consider the left-hand portion of the beam (Fig. 173 (*b*)). If P and $W_1$ are equal, there is no resultant tendency to produce vertical movement of this portion. Otherwise the stresses at the section must supply an upward or downward force S according as $W_1$ is greater or less than P. S is called the **shearing force** at the section AB, and is calculated by taking the algebraic sum of all the forces applied to either one portion or the other portion of the beam.

A common convention is to describe bending moments as positive or negative according as the beam bends as shown in Fig. 174 (*a*) or

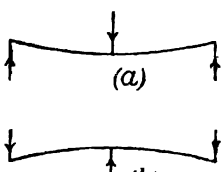



FIG. 174.—Positive and negative bending.

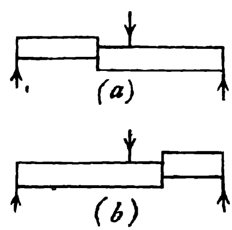



FIG. 175.—Positive and negative shear.

Fig. 174 (*b*). If the action is as shown in Fig. 175 (*a*), the shearing force is positive ; Fig. 175 (*b*) shows the action with a negative shearing force.

EXAMPLE.—A beam of 20 feet span is supported at its ends and carries a uniformly distributed load of 1 ton weight per foot length (Fig. 176 (*a*)). Find the bending moment and shearing force at a section 6 feet from the left-hand support.

The total load is 20 tons weight, and the reaction of each support is therefore 10 tons weight. Referring to Fig. 176 (*b*), it will be noted that the external forces applied to the portion of the beam lying on the left-hand side of the section are 10 tons weight acting upwards and a distributed load of 6 tons weight acting downwards. The latter may be applied at its centre of gravity, *i.e.* 3 feet from the section.

$$\text{Bending moment} = (10 \times 6) - (6 \times 3) = 60 - 18$$
$$= \underline{42} \text{ ton-feet.}$$
$$\text{Shearing force} = 10 - 6 = \underline{4} \text{ tons weight.}$$

In the same manner the bending moments and shearing forces at other sections may be calculated and the results plotted (Fig. 176 (*c*) and (*d*)).

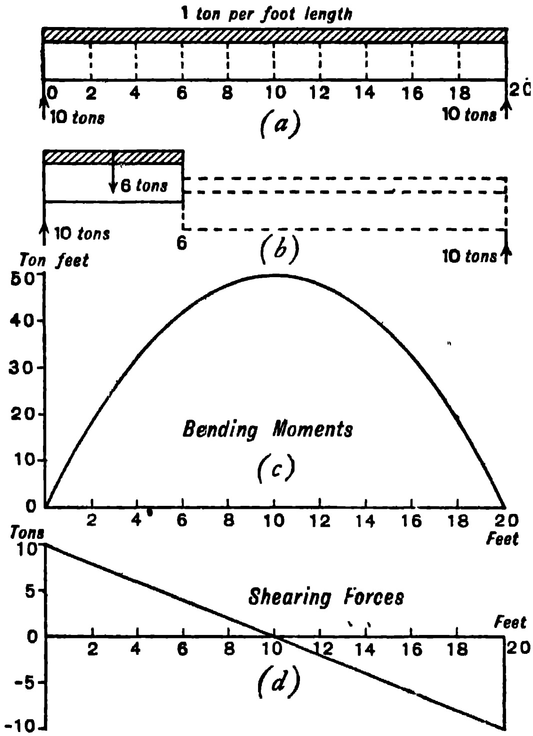



FIG. 176.—Bending moment and shearing force diagrams for a beam carrying a uniformly distributed load.

The resulting diagrams show clearly how the bending moments and shearing forces vary throughout the beam.

EXPT. 27.—**Deflection of a beam.** The apparatus employed is shown in Fig. 177, and consists of two cast-iron brackets A and B, which can be clamped anywhere to a lathe bed C, or other rigid support. The brackets have knife edges at the tops, and the test beam rests on these. A wrought-iron stirrup D, with a knife edge for resting on the beam, carries a hook E for applying the load. The deflections are measured by means of a light lever F, pivoted to a fixed support G, and attached by a fine wire at its shorter end to the stirrup; the other end moves over a fixed scale H as the beam deflects. The ratio of the lever arms may be anything from

1 : 10 to 1 : 20. The deflection produced by any load will be obtained by dividing the difference in the scale readings before and after applying the load by the ratio of the long arm to the short arm of the lever. If the test

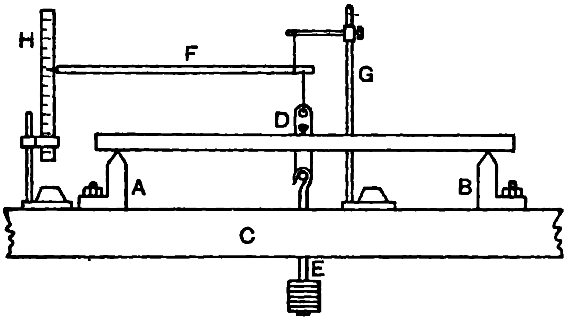



FIG. 177.—Apparatus for measuring the deflection of a beam.

beam is of timber, it is advisable to place small metal plates at *a*, *b* and *c* (Fig. 178) in order to prevent indentation of the soft material.

Arrange the apparatus as shown. Let the beam be of rectangular section; note the material and measure the span L, the breadth *b* and the

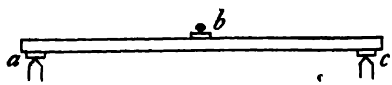



FIG. 178.

depth *d*. Apply the load at the middle of the span, and take readings as indicated in the table for a series of gradually increasing loads W. Take readings also when the load is removed step by step.

DEFLECTION TEST ON A BEAM

| Load, W | Scale reading | | Deflection |
|---|---|---|---|
| | Load increasing | Load decreasing | |
| | | | |

Plot a graph showing loads as ordinates and corresponding deflections as abscissae. A straight line graph will indicate that the deflection is proportional to the load.

The deflection of a beam is chiefly due to the longitudinal strains caused by the push and pull stresses to which the fibres of the beam are subjected. Hence the deflection is related to the value of Young's modulus of the

material. Select a point on the graph, and read off the values of W and the deflection $\Delta$ corresponding to this point. Calculate Young's modulus from

$$E=\frac{WL^3}{4\Delta bd^3}.$$

EXPT. 28.—**Deflection of a beam** (*continued*). Using the beam as in Expt. 27, perform several experiments, taking a different value of L for each experiment. In each case find the mean value of $W/\Delta$ and prove that $L^3W/\Delta$ is constant. Thus for a given load, the deflection $\Delta$ is proportional to the cube of the length of the beam.

Next turn the beam on to its edge, so that the original breadth becomes the depth and depth becomes breadth. For one particular value of L, make a series of readings of W and $\Delta$, and bearing in mind the new values of $b$ and $d$, show that $\Delta/W$ is inversely proportional to breadth × depth.[3]

## Exercises on Chapter XI

**1.** A load of 7 tons weight is hung from a vertical bar of rectangular section 2·5 inches × 1 inch. Find the tensile stress.

**2.** Find the safe pull which may be applied to a bar of rectangular section 4 inches × $\frac{5}{8}$ inch, if the tensile stress allowed is 5 tons wt. per square inch.

**3.** A pull of 15 tons weight is applied to a bar of circular section. Find the diameter of the bar if the tensile stress permitted is 6 tons wt. per square inch.

**4.** Find the safe load which can be applied to a hollow cast-iron column 6 inches external and 4·5 inches internal diameter. The compressive stress allowed is 7 tons wt. per square inch.

**5.** A shearing force of 3 tons wt. is distributed uniformly over the cross-section of a pin 1·5 inches in diameter. Find the shear stress.

**6.** A steel cylinder, 3 feet long and 5 inches in diameter, is subjected to hydrostatic stress, and the volume is found to change by 0·57 cubic inch. Find the volumetric strain.

**7.** A square steel plate 4 feet edge, plane vertical, has its lower edge fixed rigidly. Shear stress is applied and the upper edge is observed to move parallel to the lower edge through 0·02 inch. Find the shear strain.

**8.** A column, 20 feet high and having a cross-sectional area of 12 square inches, carries a load of 36 tons weight. Find the decrease in length when the load is applied. E = 29,000,000 lb. wt. per sq. inch.

**9.** A wire, 120 inches long and having a sectional area of 0·125 square inch, hangs vertically. When a load of 450 lb. weight is applied, the wire is found to stretch 0·015 inch. Find the stress, the strain, and the value of Young's modulus.

**10.** Find the tensile stress in a bolt 2·5 inches diameter when a load of 30 tons weight is applied. If $E = 30 \times 10^6$ lb. wt. per square inch, find the longitudinal strain. The original length of the bolt was 102 inches; find the extension when the load is applied.

**11.** A cast-iron bar, diameter 0·474 inch, length 8 inches, was loaded in compression, and the contraction in length caused by a gradually increasing series of loads was measured :

| Load, lb. wt. | 0 | 100 | 200 | 300 | 400 |
|---|---|---|---|---|---|
| Contraction in length, inches | 0·000 | 0·00038 | 0·00069 | 0·00105 | 0·00137 |
| Load, lb. wt. | 500 | 600 | 700 | 800 | |
| Contraction in length, inches | 0·0017 | 0·00208 | 0·0024 | 0·0027 | |

Plot a graph and find the value of Young's modulus.

**12.** A square steel plate, 6 feet edge, has one edge rigidly fixed and shear stress of 3·704 tons wt. per square inch applied to the other edges. If the modulus of rigidity is 5500 tons wt. per square inch, find the movement of the edge opposite the fixed edge.

**13.** If the bulk modulus for copper is 3300 tons wt. per square inch, find the contraction in volume of a copper sphere 10 inches in diameter when subjected to a hydrostatic stress of 0·5 ton wt. per square inch.

**14.** A specimen of steel, 0·714 inch in diameter and 7·81 inches long, had an angle of twist of 0·56 degree when a torque of 400 lb.-inches was applied. Find the value of the modulus of rigidity.

**15.** A beam, 20 feet long, rests on supports at its ends. There is a load of 2 tons weight at the middle, and other two loads of 1 ton weight each placed at points 5 feet from each support. Find the bending moment at each load ; also the shearing force at a section 6 feet from one support. Neglect the weight of the beam.

**16.** A cantilever projects 8 feet from a wall and carries a load of 400 lb. weight distributed uniformly over the length of the cantilever. Calculate the bending moments and shearing forces at sections 0, 2, 4, 6 and 8 feet from the wall. Draw diagrams of bending moments and shearing forces.

**17.** In testing a steel bar as a beam supported at the ends and loaded at the middle, it was found that a load of 10 lb. weight produced a deflection of 0·0053 inch. The beam was 1 inch broad, 1 inch deep and 40 inches span. Find the value of Young's modulus.

**18.** Discuss the nature of the forces acting on the fibres at any cross-section of a beam fixed at one end and loaded at the other.

A uniform beam, 20 ft. long, weighing 2000 lb., is supported at its ends. The beam carries a weight of 4000 lb. at a point 5 ft. from one end. Find the bending moments at the centre of the beam and at the point where the weight is supported.

**19.** A light horizontal beam AB, of length 7 feet, is supported at its ends, and loaded with weights 40 and 50 lb. at distances of 2 and 4 feet from A. Find the reactions at A and B, and tabulate the bending moment and shearing force at distances 1, 3, 5 and 7 feet from A. Draw a diagram from which can be found the bending moment at any point of the beam.

**20.** What is meant by Young's modulus? A wire of diameter 2 mm. is stretched horizontally between two rigid supports 1 metre apart. To the middle point of the wire is attached a weight of 100 gm., which causes a sag of 1·2 cm. Find the Young's modulus of the wire. (The initial tension may be taken as zero.) C.W.B., H.C.

**21.** Explain what you mean by the terms: bulk modulus and rigidity modulus.

Describe an experiment by which you would determine the rigidity modulus of a wire, stating clearly how you would calculate the result from your observations. C.W.B., H.C.

**22.** State the relation between the extension of an elastic string and its tension.

To the mid-point C of an unstretched elastic string of length $2a$, secured at its ends by two pegs A and B (also $2a$ apart), is attached another elastic string of length $a$. The other end D of this string is pulled at right angles to AB until C is displaced a distance equal to $a/10$. Find the shift of D assuming the strings to have the same modulus of elasticity.

J.M.B., H.S.C.

**23.** State Hooke's law of elasticity. A long uniform copper wire, fixed at its upper end, hangs vertically and a gradually increasing load is applied at its lower end, until the wire finally breaks. Describe (giving a diagram) how the extension of the wire varies with the load, indicating clearly on your diagram the point at which Hooke's law ceases to be applicable. Calculate the mean cross-section of a steel wire 10 metres long which increases 5 mm. in length under a load of 10 kilograms. (Young's modulus for steel $= 2 \times 10^{12}$ dynes/cm.$^2$) J.M.B., H.S.C.

# CHAPTER XII

## WORK. ENERGY. POWER.

**Work.**—**Work** is said to be done by a force when the point of application undergoes a displacement along the line of action of the force.

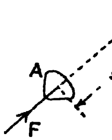


FIG. 179.—Work done by a force.

Work is measured by the product of the magnitude of the force and the displacement. Thus, if A (Fig. 179) be displaced from A to B, a distance $s$ along the line of action of the force F, then

Work done by $F = Fs$. ..................(1)

In Fig. 180 the point of application is displaced from A to B, and AB does not coincide with the direction of F. The displacement AB is equivalent to the component displacements AC and CB, which are respectively along and at right angles to the line of F. Let $s$ denote the displacement AC, then

Work done by $F = Fs = F \times AB \times \cos \alpha$. ..................(2)

The work done by F may also be calculated by the following method: In Fig. 181 take components of F, (P and Q), respectively at right angles

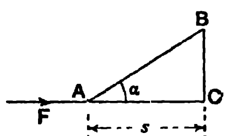


FIG. 180.

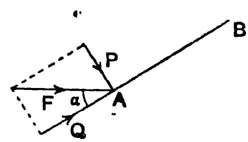


FIG. 181.

to and along AB. Q is equal to $F \cos \alpha$. P does no work during the displacement from A to B; the work is done by Q alone, and is given by

Work done $= Q \times AB = F \times \cos \alpha \times AB$, ..................(3)

which is the same result as before.

No work is done against gravity when a load is carried along a level road. This follows from the consideration that the point of application of the vertical force supporting the load moves in a horizontal plane, and therefore undergoes no displacement in the vertical line of action of the weight.

**Units of work.**—Unit work is performed when unit force produces unit displacement. The C.G.S. absolute unit of work is the **erg**, and is performed when a force of one dyne acts through a distance of one centimetre. For many purposes the **joule** is used as the practical unit of work and is equal to $10^7$ ergs. The reason for this is that when the

E.M.F. in an electrical circuit is one volt and the current maintained is one ampere the rate of working is $10^7$ ergs per second (see Chapter LXVI), so that under these conditions the rate of working is one joule per second.

On the M.K.S. system (p. 2) the joule is the absolute unit; that is, it is the unit expressed directly in terms of the metre, kilogram and second. For, 1 metre = 100 centimetres, so that the unit of acceleration or 1 metre per second per second is 100 cm. per sec. per sec. The force which would produce this acceleration in one kilogram is $1000 \times 100$ dynes, and the work done when this force acts through 1 metre or 100 cm. is $1000 \times 100 \times 100 = 10^7$ ergs or 1 joule. Thus the joule is the amount of work done when a force which produces an acceleration of 1 metre per sec. per sec. in a mass of 1 kilogram acts through a distance of 1 metre.

The British absolute unit of work is the **foot-poundal**, and is performed when a force of one poundal acts through a distance of one foot. The gravitational unit of work is the **foot-lb.**, and is performed when a force of one pound weight acts through a distance of one foot.

The dimensions of work are

$$\frac{ml}{t^2} \times l = \frac{ml^2}{t^2} \quad \text{or} \quad ml^2t^{-2}.$$

**Work done in elevating a body.**—In Fig. 182 is shown a body having a total weight W, and having its centre of gravity $G_1$ at a height H above the ground. $w_1$, $w_2$, etc., are particles situated at heights $h_1$, $h_2$, etc., above the ground. Let the body be raised so that $G_1$ moves to G′, and $w_1$, $w_2$ to $w_1'$, $w_2'$ at heights H′, $h_1'$ and $h_2'$ respectively. The work done against gravity in raising $w_1$ and $w_2$ is $w_1(h_1' - h_1)$ and $w_2(h_2' - h_2)$; hence the total work done in raising the body is given by

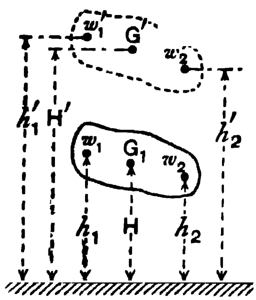



FIG. 182.—Work done in raising a body.

$$\begin{aligned} \text{Work done} &= w_1(h_1' - h_1) + w_2(h_2' - h_2) + w_3(h_3' - h_3) + \text{etc.} \\ &= (w_1h_1' + w_2h_2' + w_3h_3' + \text{etc.}) - (w_1h_1 + w_2h_2 + w_3h_3 + \text{etc.}) \\ &= \mathrm{WH'} - \mathrm{WH} \quad \text{(p. 98)} \\ &= \mathrm{W(H' - H)}. \end{aligned}$$

The work done in raising a body against the action of gravity may therefore be calculated by taking the product of the total weight of the body and the vertical height through which the centre of gravity is raised.

**Graphic representation of work.**—Since work is measured by the product of force and distance, it follows that the area of a diagram in

which ordinates represent force and abscissae represent distances will represent the work done.

If the force is uniform, the diagram is a rectangle (Fig. 183). The work done by a uniform force P acting through a distance D is P × D.

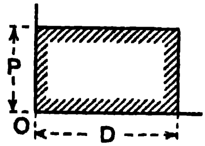


FIG. 183.—Diagram of work done by a uniform force.

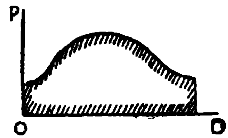


FIG. 184.—Diagram of work done by a varying force.

If unit height of the diagram represents $p$ units of force, and unit length represents $d$ units of displacement, then one unit of area of the diagram represents $pd$ units of work. If the diagram measures A units of area, then the total work done is given by $pd$A.

If the force varies (Fig. 184), the diagram of work is drawn by setting off ordinates to represent the magnitude of the force at different values of the displacement. The work done may be calculated by taking the product of the average value of the force and the displacement. Since the average height of the diagram represents the average force, and the length of the diagram represents displacement, we have, as before, the work done represented by the area of the diagram, and one unit of area of the diagram represents $pd$ units of work. The area A of the diagram may be found by means of a planimeter, or by any convenient rule of mensuration, when $pd$A will give the total work done.

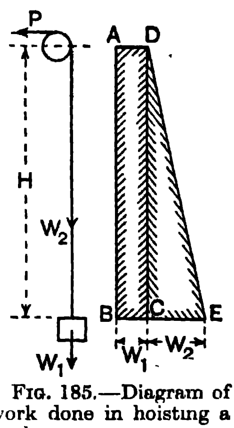


FIG. 185.—Diagram of work done in hoisting a load.

EXAMPLE.—Find the work done against gravity when a cage and load weighing $W_1$ are raised from a pit H deep by means of a rope having a weight $W_2$ (Fig. 185).

At first the pull P required at the top of the rope is $(W_1 + W_2)$, and this diminishes gradually as the cage ascends, becoming $W_1$ when the cage is at the top. The diagram of work for hoisting the cage and load alone is the rectangle ABCD, in which BC and AB represent $W_1$ and H respectively; the diagram for hoisting the rope alone is DCE, in which $W_2$ is represented by CE. From the diagrams, we have

$$\text{Total work done} = W_1H + \tfrac{1}{2}W_2H$$
$$= (W_1 + \tfrac{1}{2}W_2)H.$$

**Energy.**—A body is said to possess energy when it is capable of doing [illegible]k. A coiled spring and a flying bullet both possess energy because [illegible] performing work, the former by uncoiling, the

latter by being brought to rest. In both cases work must be done upon the body in the first instance in order to produce the condition in which the body possesses energy. For example, the uncoiled spring requires some force or couple to act upon it in order to produce the coiling, and the bullet is given its velocity by the pressure upon it of the gas produced by explosion in the gun. There is, however, one important difference between the two cases. The coiled spring possesses energy although it is at rest, and it is then said to possess **potential energy.** The bullet possesses energy on account of its motion, and this is called **kinetic energy.**

There are many forms of potential energy; for example, a raised body possesses energy on account of its position, for as the force of gravity pulls it downwards, work is done as it descends. Two chemical substances such as hydrogen and oxygen possess potential energy, for in combining heat is liberated, and this may be converted into work in the steam engine. The equivalence of heat and work will be discussed further in Chapter XXVII.

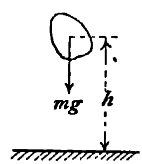


FIG. 186.—Potential energy.

Energy is measured in units of work. Thus the potential energy of a body of mass $m$ at an elevation $h$ (Fig. 186) is $mgh$, since $mgh$ absolute units of work will be done by gravitational effort whilst the body is descending.

**Conservation of energy.**—Experience shows that all energy at our disposal comes from natural sources. The **principle of the conservation of energy states that man is unable to create or destroy energy; he can only transform it from one kind into another.** For example, a labourer carrying bricks up a ladder is not creating potential energy, but is only converting some of his internal store of energy into another form. Presently rest and food will be necessary in order that his internal store of energy may be replenished. No matter what may be the form of food, it is derived ultimately from vegetation, and vegetation depends for its growth upon the light and heat of the sun. Hence the store of energy in the sun is responsible primarily for the elevation of the bricks. The student will be able to supply other examples from his own experience.

The statement that energy cannot be destroyed requires some explanation. In converting energy from one form into another, some of the energy disappears generally, so that the total energy in the new form is less than the original energy. If careful examination be made, it will be found that the missing energy has been converted into forms other than that desired, and that the total energy in the various final

forms is exactly equal to the original energy. For example, a hammer is used for driving a nail and is given kinetic energy by the operator. The hammer strikes the nail, and some of its energy is used in performing the useful work of driving the nail. The remainder is wasted in damaging the head of the nail and in the production of sound and heat. The student should accustom himself to the use of the term 'wasted energy' in preference to 'lost energy', which might lead to the idea that some energy had been destroyed.

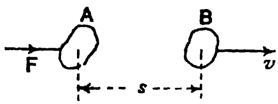


FIG. 187.—Kinetic energy.

**Kinetic energy.**—In Fig. 187 a resultant external force F acts on a mass $m$, which is at rest at A. Let the body be displaced to B through a distance $s$, and let its velocity at B be $v$. Then

$$\text{Work done by F} = \text{F}s. \qquad \qquad (1)$$

None of this work has been done against any external resistance, hence it must be stored in the body at B in the form of kinetic energy.

Hence, Kinetic energy at B $= \text{F}s = mas$.

Also, $$v^2 = 2as, \quad \text{or} \quad a = \frac{v^2}{2s}; \quad \text{(p. 26)}$$

$$\therefore \text{ Kinetic energy at B} = ms \cdot \frac{v^2}{2s}$$

$$= \frac{mv^2}{2} \text{ absolute units}, \qquad \qquad (2)$$

or, more generally, since $\text{F} = ma = m\dfrac{dv}{dt}$,

$$\text{Work} = \text{F} \cdot ds = m\frac{dv}{dt} \cdot ds = m\frac{ds}{dt} \cdot dv.$$

But, $$\frac{ds}{dt} = v, \quad \therefore \text{ F} \cdot ds = mv \cdot dv.$$

If $v_1$ is the velocity at position $s_1$ and $v_2$ the velocity at position $s_2$,

$$\text{Work in moving from } s_1 \text{ to } s_2 = \int_{s_1}^{s_2} \text{F} \cdot ds$$

$$= \int_{v_1}^{v_2} mv \cdot dv$$

$$= \tfrac{1}{2}m\Big[v^2\Big]_{v_1}^{v_2}$$

$$= \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2.$$

Thus the work done in changing the velocity from $v_1$ to $v_2$ is $\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$, and if $v_1 = 0$, the work done in giving the body velocity $v_2$ is $\frac{1}{2}mv_2^2$.

It will be noted that the result obtained for the kinetic energy is independent of the direction of motion of the body. This follows from consideration of the fact that the velocity appears to the second power in the result, which is therefore independent of the direction or sign of the given velocity. Kinetic energy is a scalar quantity.

The dimensions of kinetic energy are $ml^2/t^2$, *i.e.* kinetic energy has the same dimensions as work.

**Average resistance.**—When a body is in motion and it is desired to bring it to rest, or to diminish its speed, a force must be applied having a sense opposite to that of the velocity. In general it is not possible to state the precise value of this resistance at any instant, but the average value may be calculated from the consideration that the change in kinetic energy must be equal to the work done against the resistance. The following example illustrates this application of the principle of the conservation of energy.

EXAMPLE.—A stone weighing 8 lb. falls from the top of a cliff 120 feet high and buries itself 4 feet deep in the sand. Find the average resistance to penetration offered by the sand, and the approximate time of penetration. L.U.

In this case it is simpler to use gravitational units of force ; thus :

$$\text{Total energy available} = \text{potential energy transformed}$$
$$= 8 \times (120 + 4) = 992 \text{ foot-lb.}$$

Let $\mathsf{P}$ = the average resistance in lb. weight,

then, Work done against $\mathsf{P} = \mathsf{P} \times 4$ foot-lb. ;

$$\therefore \; 4\mathsf{P} = 992,$$
$$\mathsf{P} = \underline{248} \text{ lb. weight.}$$

Again, the velocity just before reaching the sand is given by

$$v = \sqrt{2gh} = \sqrt{64{\cdot}4 \times 120},$$
$$v = 87{\cdot}9 \text{ feet per sec.}$$

Also, Average velocity × time = distance travelled ;

$$\therefore \; \frac{87{\cdot}9}{2} \times t = 4,$$
$$t = \frac{8}{87{\cdot}9}$$
$$= \underline{0{\cdot}091} \text{ second.}$$

**Virtual work.**—In many problems a solution may be found by considering the work done at various points in a system when a slight displacement is supposed to be produced without any external force acting on the body, the displacement being so small that the condition of equilibrium of the body is unaltered. The various forces acting at different points of the body during this small displacement perform work, but if the forces acting upon the body are in equilibrium, the

algebraic sum of these amounts of work is zero. This is known as the principle of **virtual work.**

As an example consider the problem of finding the resultant of two parallel forces (see p. 87). If the two parallel forces P and Q have resultant equal and opposite to R (Fig. 188 (*a*)), let the body upon

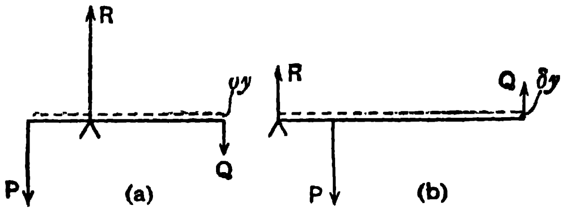



FIG. 188.—Virtual work.

which they act be displaced in the direction of the forces by the small amount $\delta y$. Then the work done on account of P is $P \, . \, \delta y$, of Q is $Q \, . \, \delta y$, and on account of R is $R \, . \, \delta y$. By the principle of virtual work

$$P \, . \, \delta y + Q \, . \, \delta y - R \, . \, \delta y = 0 \, ;$$

$$\therefore \; R = P + Q.$$

In case (*b*),

$$P \, . \, \delta y - Q \, . \, \delta y - R \, . \, \delta y = 0 \, ;$$

$$\therefore \; R = P - Q.$$

Again, let the body be rotated through a small angle $\delta\theta$ about O (Fig. 189 (*a*)), the point, wherever it may be, at which the resultant

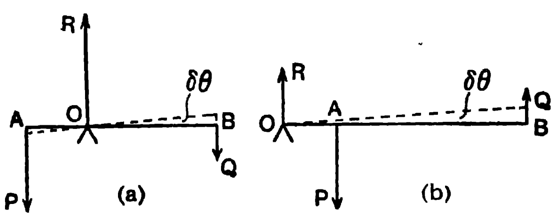



FIG. 189.—Virtual work.

acts. Then O is not displaced by the rotation and there is no corresponding work done. But the point of application of P moves through distance $AO \, . \, \delta\theta$ and the work done is $P \, . \, AO \, . \, \delta\theta$. Similarly for Q, the work done is $Q \, . \, OB \, . \, \delta\theta$ in the opposite direction to that done on account of P. Thus, by the principle of virtual work,

$$P \, . \, AO \, . \, \delta\theta - Q \, . \, OB \, . \, \delta\theta = 0,$$

or,

$$P \times OA = Q \times OB,$$

a result already seen on p. 88.

Similarly for Fig. 189 (*b*), work for P is $P \cdot OA \cdot \delta\theta$, and for Q, $Q \cdot OB \cdot \delta\theta$, so that again

$$P \times OA = Q \times OB.$$

**Work done by a couple.**—Let a couple PQ (Fig. 190) act on a body, the moment of the couple being $P \times a$. If the couple rotates the body through angle $\theta$ about any point such as O, the point where A, the foot of the perpendicular from O on P, moves through the arc AB of length $\theta \cdot OA$. The work done by P is therefore $P \cdot \theta \cdot OA$. Similarly, the work done by Q is $Q \cdot \theta \cdot OC$, and the total work done is

$$P \cdot \theta \cdot OA + Q \cdot \theta \cdot OC.$$

But $P = Q$ and $OA + OC = a$,

$\therefore$ work done is $Pa\theta = \text{couple} \times \theta$,

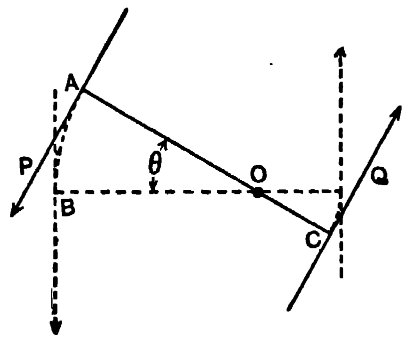



FIG. 190.—Couple performing work.

and this is independent of the position of the point O, about which rotation occurs.

For a complete rotation $\theta = 2\pi$, and for $n$ complete rotations $\theta = 2\pi n$.

$\therefore$ Work done for $n$ complete turns $= 2\pi n$ (couple).

**Power or rate of working.**—When work is done, the rate at which this is performed is of great importance. This **rate of working** or **power** may be expressed in absolute units as ergs per second or foot-poundals per second. It may also be expressed in practical units as joules per second, or again in foot-pounds per minute. The former unit is called the **watt**. Thus the rate of working in watts is the number of joules performed per second, and since 1 joule $= 10^7$ ergs, the rate of working in watts is obtained by expressing the rate of working in ergs per second and dividing by $10^7$. It will be seen in Chapter LXVI that the rate of working in an electric circuit expressed in watts is the product of volts and amperes.

On the British system the unit rate of working is 33,000 foot-pounds per minute, and is called the **horse-power**, generally written H.P. By converting from one system to the other it may be found that one horse-power is equivalent to 746 watts.

When energy is supplied by means of electric current it is charged by the supplier at so much per Board of Trade unit of electrical energy. This unit is the energy supplied when a rate of working of 1000 watts is continued for one hour and is called the **kilowatt-hour**. It is therefore $1000 \times 60 \times 60 = 3{\cdot}6 \times 10^6$ joules or $3{\cdot}6 \times 10^{13}$ ergs.

**Example.**—A shaft is driven by a belt on a pulley of diameter 1' 6". The tension in the belt is 800 lb.-wt. on one side and 50 lb.-wt. on the other. If the shaft and pulley make 120 revolutions per minute, what is the power transmitted in horse-power and in watts?

$$\text{Couple acting on shaft} = (800 - 50)(0{\cdot}75) \text{ lb.-wt.-foot.}$$
$$\text{Work per revolution} = 2\pi \times 750 \times (0{\cdot}75) \text{ ft.-lb.}$$
$$\text{Work per minute} = 2\pi \times 750 \times 0{\cdot}75 \times 120 \text{ ft.-lb.}$$
$$\text{Horse-power} = \frac{2\pi \times 750 \times 0{\cdot}75 \times 120}{33000}$$
$$= \underline{12{\cdot}85.}$$
$$\text{Watts} = 12{\cdot}85 \times 746$$
$$= \underline{8293.}$$

## Exercises on Chapter XII

**1.** A load of 3 tons weight is raised from the bottom of a shaft 600 ft. deep. Calculate the work done.

**2.** In Question 1 the wire rope used for raising the load weighs 12 lb. per yard. Find the total work done.

**3.** Calculate the work done in hauling a loaded truck, weight 12 tons, along a level track one mile long. The resistances to motion are 11 lb. weight per ton weight of truck.

**4.** A well is 100 feet deep and 10 feet in diameter, and is full of water (62·5 lb. weight per cubic foot). Calculate the work done in pumping the whole of the water up to ground level.

**5.** A pyramid of masonry has a square base of 40 feet side and is 30 feet high. If masonry weighs 150 lb. per cubic foot, how much work must be done against gravity in placing the stones into position?

**6.** The head of a hammer has a mass of 2 pounds and is moving at 40 feet per second. Find the kinetic energy.

**7.** A ship having a mass of 15,000 tons has a speed of 20 knots (1 knot = 6080 feet per hour). What is the kinetic energy in foot-tons? If the ship is brought to rest in a distance of 0·5 mile, what has been the average resistance?

**8.** A train having a mass of 200 tons is travelling at 30 miles per hour on a level track. Find the average pull in tons weight which must be applied in order to increase the speed to 40 miles per hour while the train travels a distance of 3000 feet. Neglect frictional resistances.

**9.** A bullet has a mass of 0·03 pound, and is fired with a velocity of 2400 feet per second into a sand-bank. If the bullet penetrates a distance of 3 feet, what has been the average resistance?

**10.** A horse walks at a steady rate of 3 miles an hour along a level road and exerts a pull of 80 lb. weight in dragging a cart. What horse-power is he developing?

**11.** Find the useful horse-power used in pumping 5000 gallons of water per minute from a well 40 feet deep to the surface of the water. Supposing 40 per cent. of the horse-power of the engine driving the pump is wasted, what is the horse-power of the engine?

**12.** A pump raises 6·2 cubic feet of water per second to a height of 7 feet; how much horse-power must be supplied if 55 per cent. is wasted? The pump is driven by an electro-motor, and current is supplied at 200 volts. How many amperes of current must be supplied to the motor assuming that the motor wastes 15 per cent. of the energy supplied to it?

**13.** A belt runs at 2000 feet per minute. The pulls in the straight portions are 200 and 440 lb. weight respectively. What horse-power is being transmitted?

**14.** A belt transmits 60 horse-power to a pulley. If the pulley is 16 inches in diameter and runs at 263 revolutions per minute, what is the difference of the tensions on the two straight portions?

**15.** A pile-driver weighing 3 cwt. falls from a height of 20 feet on a pile weighing 15 cwt.: if there is no rebound, calculate how far the pile will be driven against a constant resistance equal to the weight of 30 cwt.?

**16.** Define energy, kinetic energy, and potential energy; and show that when a particle of mass $m$ is dropped from a height $h$, the sum of its kinetic and potential energies at any instant during motion is constant and equals $mgh$.

**17.** A motor-car develops 20 horse-power in travelling at a speed of 40 miles per hour up a hill having a slope 1 in 50. If the frictional resistance is 80 lb. wt. per ton weight of car, find the weight of the car, and the speed it could reach on the level, supposing the horse-power developed and the resistance to be unaltered.

**18.** Explain the difference between the momentum and the kinetic energy of a moving body. Two bodies, A and B, weigh 10 lb. and 40 lb. respectively. Each is acted upon by a force equal to the weight of 5 lb. Compare the times the forces must act to produce in each of the bodies (*a*) the same momentum, (*b*) the same kinetic energy.

**19.** Define work and power, and give their dimensions in terms of the fundamental units of mass, length and time. The maximum speed of a motor-van weighing 3 tons is 12 miles an hour on a level road, but drops to 5 miles an hour up an incline of 1 in 10. Assuming resistances per ton to vary as the square of the velocity, find the horse-power of the engine. L.U.

**20.** If a train moving at 60 miles per hour can be pulled up by its brakes in $\frac{1}{4}$ mile, from what speed could it be pulled up in 55 yards? What is the time taken in coming to a standstill in each case? C.W.B., H.C.

**21.** Prove that the kinetic energy of two particles of masses $m_1$ and $m_2$ moving in any manner in a plane is equal to

$$\tfrac{1}{2}(m_1+m_2)v^2+\tfrac{1}{2}m_1m_2u^2/(m_1+m_2),$$

where $v$ is the velocity of the centre of gravity of the two particles and $u$ is the velocity of one particle relative to the other. J.M.B., H.S.C.

**22.** A train of 375 tons is being drawn up an incline of 1 in 168 (slope $=\sin^{-1}\frac{1}{168}$) with a uniform acceleration of $\frac{1}{4}$ foot per second per second by two engines, each weighing 75 tons. If the frictional resistance is 16 lb. weight per ton and both engines are working at the same rate, show that, when the speed of the train is 30 miles per hour, each engine is working at 983·5 H.P. ($g=32$.) Also find the pulls in the couplings (1) between the second engine and the train, (2) between the engines. J.M.B., H.S.C.

**23.** A 10 horse-power electric motor is getting up speed from rest against a resisting couple whose moment is 320 poundal-feet. Find the maximum angular velocity of the rotating part and the time taken to attain it, if the acceleration is constant and equal to 40 radians per second per second.
J.M.B., H.S.C.

# CHAPTER XIII

## FRICTION

**Friction.**—In practice much energy is wasted in overcoming frictional resistances, and the general laws of friction should be understood by the student.

When two bodies are pressed together it is found that there is a resistance offered to the sliding of one upon the other. This resistance is called the **force of friction.** The force which friction applies to a body always acts in such a direction as to maintain the state of rest, or to oppose the motion of the body.

Let two bodies A, B (Fig. 191 (*a*)) be pressed together, and let the mutual force perpendicular to the surfaces in contact be R. Let B be fixed, and let a force P, parallel to the surfaces in contact, be applied (Fig. 191 (*b*)). If P is not large enough to produce sliding, or if sliding with steady speed takes place, B will apply to A a frictional force F equal and opposite to P. The force F may have any value lower than a certain maximum, which depends on the magnitude of R and on the nature and condition of the surfaces in contact. If P is less than the maximum value of F, sliding will not occur; sliding will be on the point of occurring when P is equal to the maximum possible value of F. It is found that the frictional resistance offered, after steady sliding conditions have been attained, is less than that offered when the body is on the point of sliding.

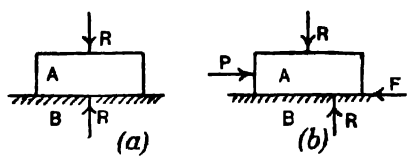


FIG. 191.—Force of friction.

Let $F_s$ = the frictional resistance when the body is on the point of sliding.

$F_k$ = the frictional resistance when steady sliding has been attained.

R = the perpendicular force between the surfaces in contact.

These forces should all be stated in the same units. Then

$$\mu_s = \frac{F_s}{R}; \quad \mu_k = \frac{F_k}{R}.$$

$\mu_s$ and $\mu_k$ are called respectively the **static and kinetic coefficients of friction.**

**Friction of dry surfaces.**—Owing to the great influence of apparently trifling alterations in the state of the rubbing surfaces, it is not possible

to predict with any pretence at accuracy what the frictional resistance will be in any given case. For this reason the proper place to study friction is in a laboratory having suitable apparatus. For dry clean surfaces the following general laws are complied with roughly :

**The force of friction is proportional to the perpendicular force between the surfaces in contact, and is independent of the extent of these surfaces and of the speed of rubbing, if moderate.** It therefore follows that the **kinetic coefficient of friction for two given bodies is practically constant for moderate pressures and speeds.** Experiments on the static coefficient of friction are not performed easily ; roughly, this coefficient is constant for two given bodies.

COEFFICIENTS OF FRICTION

*(Average values)*

| | |
|---|---|
| Metal on metal, dry | 0·2 |
| Metal on wood, dry | 0·6 |
| Wood on wood, dry | 0·2 to 0·5 |
| Leather on iron | 0·3 to 0·5 |
| Leather on wood | 0·3 to 0·5 |
| Stone on stone | 0·7 |
| Wood on stone | 0·6 |
| Metal on stone | 0·5 |

The average values given in this table should be employed only in the absence of more definite experimental values for the bodies concerned.

EXPT. 29.—**Determination of the kinetic coefficient of friction.** Set up a board AB (Fig. 192) as nearly horizontal as possible, and arrange a slider C (which can be loaded to any desired amount) with a cord, pulley and scale-pan, so that the horizontal force P required to overcome the frictional resistance may be measured. Weigh the slider, and let its weight, together with the load placed on it, be called W. The perpendicular force between the surfaces in contact will be equal to W. Weigh the scale-pan, and let its weight, together with the weights placed in it in order to secure steady sliding, be called P. P and F will be equal ; hence

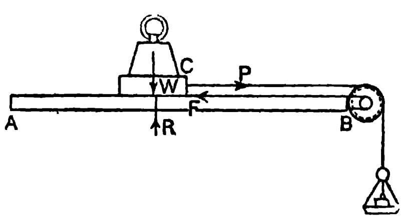



FIG. 192.—Friction of a slider.

$$\text{Kinetic coefficient of friction} = \frac{P}{W}.$$

It is necessary to assist the slider to start by tapping the board. The rubbing surfaces should be clean and free from dust. More consistent results can be obtained from surfaces which have been freshly planed.

Make a series of about ten experiments with gradually increasing loads. Plot P and W; the plotted points will lie approximately on a straight line. Draw the straight line which best fits the points; select one point on the graph, and read the values of P and W for it; let these values be $P_1$ and $W_1$, then

$$\text{Average kinetic coefficient of friction} = \frac{P_1}{W_1}.$$

The materials of which the slider and board are made should be stated, and, if these are timber, whether rubbing has been with the grain or across the grain of the wood.

**Friction on an inclined plane.**—In Fig. 193, XZ is an inclined board which has been arranged so that a block A just slides down with steady speed. Let *ca* represent the weight of the block; by means of the parallelogram of forces *cbad*, find the components Q and P of W, respectively perpendicular and parallel to XZ. The board applies a frictional force F to the block in a direction coinciding with the surface of the board and contrary to the motion of the block, *i.e.* up the plane. As there is no acceleration, P and F are equal. The plane also exerts on the block a force R, equal and opposite to Q. R is the normal or perpendicular force between the surfaces in contact. Hence, by the definition (p. 153),

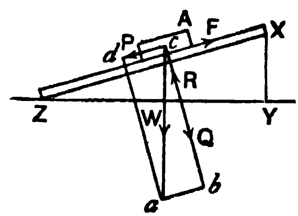



FIG. 193.—Coefficient of friction determined by inclining the board.

$$\text{Kinetic coefficient of friction} = \frac{F}{R} = \frac{P}{Q}.$$

Since *cb* and *ca* are perpendicular to XZ and ZY respectively, it follows that the angles *acb* and XZY are equal. Hence

$$Q = W\cos acb = W\cos XZY,$$
$$P = W\sin acb = W\sin XZY;$$
$$\therefore\ \mu_k = \frac{P}{Q} = \frac{W\sin XZY}{W\cos XZY}$$
$$= \tan XZY.$$

The angle XZY is called the **angle of sliding friction.**

EXPT. 30.—**Determination of $\mu_k$ from the angle of sliding friction.** Use the same board and slider as in Expt. 29. Raise one end of the board until, with assistance in starting, the slider travels down the inclined plane with constant speed. Measure the angle of inclination of the plane, or measure its height and base, and so obtain the tangent of the angle of sliding friction; this will give $\mu_k$. Place a weight on the slider, and ascertain if

the block will still slide with steady speed. Compare the result with that obtained in Expt. 29.

**Resultant reaction between two bodies.**—In Fig. 194 is shown a block A resting on a horizontal table BC. The weight W of the block acts in a line normal to BC. Let a horizontal force $P_1$ be applied to the block; $P_1$ and W will have a resultant $R_1$. For equilibrium the table must exert a resultant force on the block equal and opposite to $R_1$ and in the same straight line. Let this force be $E_1$, cutting BC in D. $E_1$ may be resolved into two forces, Q perpendicular to BC, and $F_1$ along BC. Let $\phi_1$ be the angle which $E_1$ makes with GD. Then

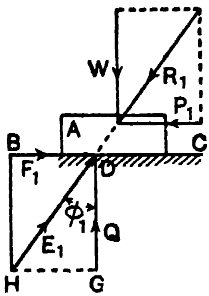


FIG. 194.—Friction

$$\frac{F_1}{Q}=\frac{HG}{GD}=\tan\phi_1.$$

Now, when $P_1$ is zero, $\phi_1$ and hence $\tan\phi_1$ will also be zero, and Q will act in the same line as W. $\phi_1$ will increase as $P_1$ increases, and will reach a maximum value when the block is on the point of slipping. It is evident that Q will always be equal to W. Let $\phi$ be the value of the angle when the block just slips, and let F be the corresponding value of the frictional force; then

$$\text{Static coefficient of friction}=\mu_s=\frac{F}{Q}=\tan\phi.$$

$\phi$ is called the **friction angle** or the **limiting angle of resistance**; when steady sliding has been attained, $\phi$ is lower in value and is called, as noted above, the angle of sliding friction.

It is evident from Fig. 194 that $P_1$ and $F_1$ are always equal (assuming no sliding, or sliding with constant speed); W and Q are also always equal. These forces form couples having equal opposing moments, and so balance the block. It will be noted that D, the point through which Q acts, does not lie in the centre of the rubbing surface unless $P_1$ is zero. The effect is partially to relieve the normal pressure near the right-hand edge of the block and to increase it near the left-hand edge. With a sufficiently large value of $\mu_s$, and by applying P at a large enough height above the table, the block can be made to overturn instead of sliding.

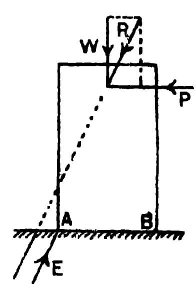


FIG. 195.—Condition that block may overturn.

In Fig. 195 the resultant R of P and W may fall outside the base AB before sliding begins. Hence E, which must act on AB, cannot act in the same line as R, and the block will

overturn. For overturning to be impossible, R must fall within AB.

EXAMPLE.—A block of weight W slides steadily on a plane inclined at an angle $\alpha$ to the horizontal under the action of a force P. Find the values of P in the following cases :

(*a*) P is horizontal and the block slides upwards.
(*b*) P is horizontal and the block slides downwards.
(*c*) P is parallel to the plane and the block slides upwards.
(*d*) P is parallel to the plane and the block slides downwards.

**Case (a).**—In Fig. 196 (*a*) draw AN perpendicular to the plane; the angle between W and AN is equal to $\alpha$. Draw AC, making with AN an

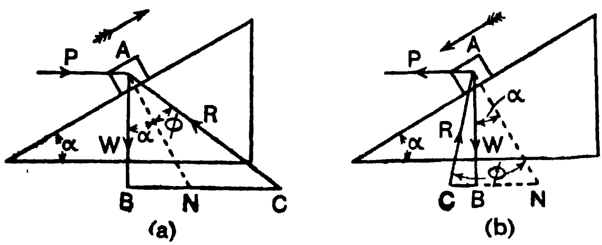



FIG. 196.—Friction on an incline ; P horizontal.

angle $\phi$ equal to the angle of sliding friction ; the resultant reaction R of the plane acts in the line CA, and ABC is the triangle of forces for W, P and R. Let $\mu$ be the kinetic coefficient of friction, then

$$\frac{P}{W}=\frac{BC}{AB}=\tan(\alpha+\phi);$$

$$\therefore\ P=W\left(\frac{\tan\alpha+\tan\phi}{1-\tan\alpha\tan\phi}\right)=W\left(\frac{\tan\alpha+\mu}{1-\mu\tan\alpha}\right). \quad\ldots\ldots\ldots\ldots(1)$$

**Case (b).**—The construction is shown in Fig. 196 (*b*), and is made as directed under Case (*a*), excepting that R acts on the other side of AN. The triangle of forces is ABC.

$$\frac{P}{W}=\frac{BC}{AB}=\tan(\phi-\alpha);$$

$$\therefore\ P=W\left(\frac{\tan\phi-\tan\alpha}{1+\tan\alpha\tan\phi}\right)=W\left(\frac{\mu-\tan\alpha}{1+\mu\tan\alpha}\right). \quad\ldots\ldots\ldots\ldots(2)$$

It will be noticed in this case that, if $\phi$ is less than $\alpha$, the block will slide down without the necessity for the application of a force P. Rest is just possible, unaided by P, if $\alpha$ and $\phi$ are equal.

**Case (c).**—The required construction is shown in Fig. 197 (*a*) ; the triangle of forces is ABC.

$$\frac{P}{W}=\frac{BC}{AB}=\frac{\sin BAC}{\sin ACB}=\frac{\sin(\alpha+\phi)}{\sin(90°-\phi)}=\frac{\sin\alpha\cos\phi+\cos\alpha\sin\phi}{\cos\phi};$$

$$\therefore\ P=W(\sin\alpha+\cos\alpha\tan\phi)=W(\sin\alpha+\mu\cos\alpha). \quad\ldots\ldots\ldots\ldots(3)$$

**Case (d).**—Referring to Fig. 197 (*b*), we have

$$\frac{P}{W}=\frac{BC}{AB}=\frac{\sin BAC}{\sin ACB}=\frac{\sin(\phi-\alpha)}{\sin(90^\circ-\phi)}$$

$$=\frac{\sin\phi\cos\alpha-\cos\phi\sin\alpha}{\cos\phi};$$

$$\therefore\ P=W(\tan\phi\cos\alpha-\sin\alpha)=W(\mu\cos\alpha-\sin\alpha).\ \ \ldots\ldots\ldots\ldots(4)$$

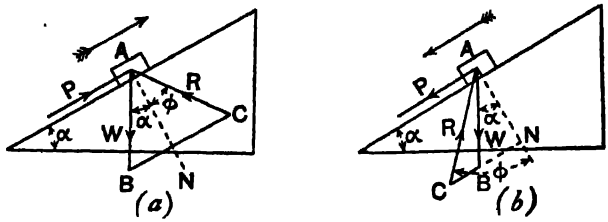



FIG. 197.—Friction on an incline ; P parallel to the incline.

**Friction of a rope coiled round a post.**—When a rope is coiled round a cylindrical post, slipping will not occur until the pull applied to one end is considerably greater than that applied to the other end. This is owing to the friction between the rope and the post having to be overcome before slipping can take place. As the frictional resistance is distributed throughout the surface in contact, the pull in the rope will diminish gradually from a value $T_1$ at one end, to $T_2$ at the other end.

In Fig. 198 the rope embraces the arc ACF, and this arc has been divided into equal arcs AB, BC, etc., subtending equal angles $\alpha$ at the centre of the post. Since these arcs are all equal, it is reasonable to suppose that the ratios of the tensions in the rope at the beginning and end of each arc are equal, *i.e.*

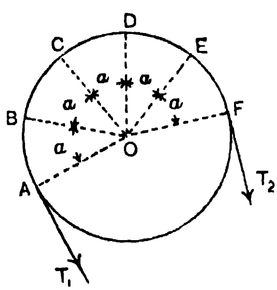



FIG. 198.—Tensions in the rope at different parts of the arc of contact.

$$\frac{T_1}{T_B}=\frac{T_B}{T_C}=\frac{T_C}{T_D}=\frac{T_D}{T_E}=\frac{T_E}{T_2}.\ \ \ldots\ldots\ldots\ldots(1)$$

This assumes that the surfaces are such that the value of the coefficient of friction is the same throughout, and the result is confirmed approximately by experiment.

A more exact expression may be found by considering one of the sections, say CD (Fig. 198), to be very short and to subtend a small angle $d\theta$ at O. If T is the tension in the rope at this place there is a force T at each end of CD, the resultant of the two pressing the rope on to the post. The two forces are inclined to each other at an angle $(2\pi-d\theta)$, and therefore have a resultant

$$\sqrt{T^2+T^2+2T^2\cos(2\pi-d\theta)}=T\sqrt{2-2\cos d\theta}\quad \text{(p. 73)}.$$

Now, $$\sin^2\frac{d\theta}{2}=\frac{1-\cos d\theta}{2};$$

$$\therefore\ 4\sin^2\frac{d\theta}{2}=2-2\cos d\theta,$$

and, $$\text{resultant force}=2\mathrm{T}\sin\frac{d\theta}{2}.$$

Since $d\theta$ is very small, $\sin\frac{d\theta}{2}=\frac{d\theta}{2}$,

and resultant force normal to post $=\mathrm{T}\,.\,d\theta$.

$$\text{Frictional force}=\mu\mathrm{T}\,.\,d\theta.$$

This frictional force implies a small difference of tension $d\mathrm{T}$ in the rope at the two ends of the section.

That is, $$d\mathrm{T}=\mu\mathrm{T}\,.\,d\theta \quad\text{or}\quad \mu\,.\,d\theta=\frac{d\mathrm{T}}{\mathrm{T}}.$$

If T is $\mathrm{T}_1$ at $\mathrm{A}_1$ where $\theta=0$ and $\mathrm{T}_2$ at F where $\theta$ is the angle ACF,

$$\mu\int_0^\theta d\theta=\int_{\mathrm{T}_1}^{\mathrm{T}_2}\frac{d\mathrm{T}}{\mathrm{T}},$$

$$\mu\theta=\Big[\log_\epsilon \mathrm{T}\Big]_{\mathrm{T}_1}^{\mathrm{T}_2}$$

$$=\log_\epsilon\frac{\mathrm{T}_2}{\mathrm{T}_1},$$

which may be expressed as $$\frac{\mathrm{T}_2}{\mathrm{T}_1}=\epsilon^{\mu\theta}$$

or, $$\mathrm{T}_2=\mathrm{T}_1\epsilon^{\mu\theta}.$$

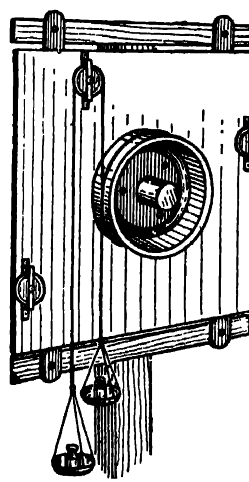


Fig. 199.—Apparatus for experiments on the friction of a cord coiled on a drum.

Expt. 31.—**Friction of a cord coiled round a post.** The apparatus shown in Fig. 199 enables the tensions to be found for angles of contact differing by 90°. Weigh the scale-pans. Put equal loads in each pan, then increase one load until steady slipping occurs. Evaluate $\mathrm{T}_1$ and $\mathrm{T}_2$, and repeat the experiment with a different angle of contact. The following is a record of an actual experiment, using a silk cord on a pine post.

An Experiment on Slipping

| Angle of lap | $\mathrm{T}_1$ lb. wt. descending | $\mathrm{T}_2$ lb. wt. ascending | Experimental ratio $\frac{\mathrm{T}_2}{\mathrm{T}_1}$ | Calculated ratio $\frac{\mathrm{T}_2}{\mathrm{T}_1}$ |
|---|---|---|---|---|
| 90° | 0·397 | 0·29 | 0·73 | 0·73 |
| 180° | 0·56 | 0·29 | 0·518 | 0·533 |
| 270° | 0·79 | 0·29 | 0·367 | 0·39 |
| 360° | 1·1 | 0·29 | 0·263 | 0·28 |

The last column is obtained as follows: Taking the first ratio of $T_2/T_1 = 0{\cdot}73$ for 90° lap, the ratio for 180° lap from (1) above would be $0{\cdot}73 \times 0{\cdot}73 = 0{\cdot}533$; the ratio for 270° is $0{\cdot}73^3 = 0{\cdot}39$; and the ratio for 360° is $0{\cdot}73^4 = 0{\cdot}28$. These calculated and experimental values show fair agreement, remembering the assumptions that have been made regarding the constancy of the coefficient of friction.

**Horse-power transmitted by a belt.**—A belt drives the pulley, round which it is wrapped, by reason of the frictional resistance between the surfaces in contact. The pulls in the straight parts of the belt differ in magnitude by an amount equal to the total frictional force round the arc of contact. Hence the pull $T_1$ (Fig. 200) in one straight portion is greater than $T_2$ in the other straight portion. Let $T_1$ and $T_2$ be stated in lb. weight, and let the speed of the belt be $V$ feet per minute; then, since $T_1$ is assisting the motion and $T_2$ is opposing it, we have

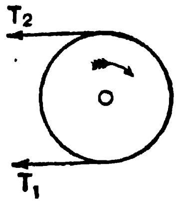



FIG. 200.

$$\text{Work done per minute} = (T_1 - T_2)V \text{ foot-lb.}$$

Hence, $$\text{Horse-power transmitted} = \frac{(T_1 - T_2)V}{33{,}000}.$$

## Exercises on Chapter XIII

**1.** A horizontal force of 8 lb. weight can keep a load weighing 30 lb. in steady motion along a horizontal table. What is the coefficient of friction? What is the minimum inclination of the table to the horizontal if the block is just able to slide steadily on it?

**2.** A block of oak rests on an oak plank 8 feet long. To what height must one end of the plank be raised before slipping will occur? The coefficient of friction is 0·45.

**3.** A block weighing $W$ lb. is dragged along a level table by a force $P$ lb. weight acting at a constant angle $\theta$ to the horizontal. The coefficient of friction is 0·25. Take successive values of $\theta = 0$, 15, 30, 45, 60 and 75 degrees, and calculate in terms of $W$ (*a*) the values of $P$, (*b*) the work done in dragging the block a distance of 1 foot. Plot graphs showing the relation of $P$ and $\theta$, and the relation of the work done and $\theta$.

**4.** A block weighs $W$ lb. and is pushed up an incline making an angle $\theta$ with the horizontal by a force $P$ lb. weight which acts in a direction parallel to the incline. The coefficient of friction is 0·25. Find in terms of $W$ (*a*) the values of $P$, (*b*) the work done in raising the block through a vertical height of one foot, in each case taking successive values of $\theta = 0$, 15, 30, 45, 60, 75 and 90 degrees. Plot graphs of $P$ and $\theta$, and of the work done and $\theta$.

**5.** Answer Question 4 if $P$ is horizontal. At what value of $\theta$ does $P$ become infinite?

**6.** A block slides down a plane inclined at 45° to the horizontal. If the coefficient of friction is 0·2, what will be the acceleration?

**7.** When a rope is coiled 180 degrees round a post, it is found that slipping occurs when one end is pulled with a force of 30 lb. weight and the other end with a force of 50 lb. weight. Supposing that the force of 30 lb. weight remains unchanged, and that three complete turns are given to the rope round the post, what force would just cause slipping?

**8.** A cyclist always works at the rate of $\frac{1}{10}$ H.P., and rides at 12 miles an hour on level ground and 10 miles an hour up an incline of 1 in 120. If the man and his machine weigh 150 lb., and the resistance on a level road consists of two parts, one constant and the other proportional to the square of the velocity, show that, when the velocity is $v$ miles per hour, the resistance is $\frac{5}{352}(76+v^2)$ lb. wt. Find also the slope up which he would travel at the rate of 8 miles per hour. L.U.

**9.** Explain what is meant by (1) the coefficient of friction, (2) the angle of friction.

A window curtain weighing 4 lb. hangs by 6 equidistant thin rings from a curtain rod in such a way that the weight is equally distributed between the rings. If the coefficient of friction is 0·6, and the rings are 6 inches apart, find the work done in drawing the curtain back to the position of the end ring. L.U.

**10.** A body having a mass of 20 pounds is placed on a rough horizontal table, and is connected by a horizontal cord passing over a pulley at the edge with a body having a mass of 10 pounds hanging vertically. If the coefficient of friction between the body and the table be 0·25, find the acceleration of the system and the pull in the cord.

**11.** Explain what is meant by the 'angle of friction'. If a body be placed on a rough horizontal plane, show that no force, however great, applied towards the plane at an angle with the normal less than the angle of friction, can push the body along the plane.

A uniform circular hoop is weighted at a point of the circumference with a mass equal to its own. Prove that the hoop can hang from a rough peg with any point of its circumference in contact with the peg, provided that the angle of friction exceeds 30°.

**12.** A ladder of length $2a$ leans against a perfectly smooth wall, the ground being slightly rough. The weight of the ladder is $w$; and its centre of gravity is at its middle point. The inclination to the vertical is gradually increased till the ladder begins to slip. The inclination is then further increased, and the ladder is prevented from slipping by the smallest possible horizontal force applied at the foot. Find the magnitude of this force if $\mu$ is the coefficient of friction and $\theta$ the final inclination to the vertical.

**13.** State the laws of sliding friction. A rectangular block with a square base of 4 inches edge rests on a rough table. A force is applied to it perpendicular to one of the vertical faces, and it is found that if the point of application is less than 5 inches above the table the block slides, if higher it tips over. Find the coefficient of friction. C.W.B., H.C.

**14.** A uniform ladder, 20 ft. long and weighing 30 lb., rests against a smooth vertical wall with its base on a rough floor and 4 ft. from the wall. If the coefficient of friction between the ladder and the floor be $\frac{1}{8}$, how far along the ladder will a man weighing 150 lb. climb before the ladder slips from under him? C.W.B., H.C.

**15.** A body rests upon an inclined plane and will just slide down the plane when the slope of the plane is 30°. Calculate the acceleration of the body down the plane when the slope is increased to 60°. ($g = 980$/cm./sec.$^2$) L.U.H.S.C.

# CHAPTER XIV

## SIMPLE MACHINES

**Machines.**—A **machine** is an arrangement designed for the purpose of taking in energy in some definite form, modifying it, and delivering it in a form more suitable for the purpose in view.

There is a large class of machines designed for the purpose of raising loads ; many of these machines can be used for experimental work in laboratories. The **crab** shown in Fig. 201 is an example. The rope to which the load W is attached is wound round a cylindrical barrel A. The machine is driven generally by hand by means of handles. For the purposes of experiment, the handles have been removed and a wheel D substituted. D is rotated by means of a cord and weights placed in a scale-pan at P, and drives the barrel by medium of the toothed wheels C and B.

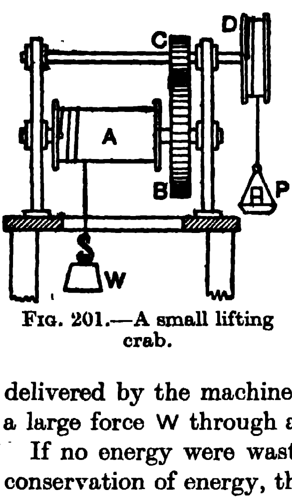


FIG. 201.—A small lifting crab.

Energy is supplied to this machine by means of a comparatively small force P acting through a large distance, and is delivered by the machine in the form of the work done in overcoming a large force W through a small distance.

If no energy were wasted in a machine, it would follow, from the conservation of energy, that the energy supplied must be equal to the energy delivered by the machine. Thus, referring to Fig. 201,

Work done by P = Work done on W.

This statement is generally referred to as the **principle of work**, and requires modification for actual machines, in which there is always some energy wasted. Actually, the energy supplied is equal to the sum of the energy delivered by the machine and the energy wasted. The investigation of frictional resistances in the various kinds of lubricated rubbing surfaces of machines is beyond the scope of this book. Usually, however, it is the determination of the total waste of energy in the machine which is of importance, and experiments having this object are performed easily in the case of simple machines used for raising loads.

**Some definitions regarding machines.**—In Fig. 202 is shown an outline diagram of the crab illustrated in Fig. 201. Let W be raised through a height $h$ while P descends through a height H, H and $h$ being in the same units. The **velocity ratio** of the machine is defined as the ratio of the distance moved by P to the distance moved by W in the same time, or

$$\text{Velocity ratio} = V = \frac{H}{h}. \quad \ldots\ldots\ldots (1)$$

H and $h$ may be obtained by direct measurement, or they may be calculated from known dimensions of the parts of the machine.

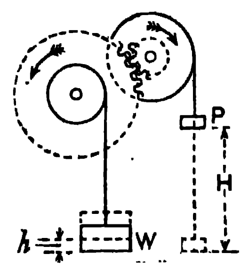


FIG. 202.—Outline diagram of an experimental crab.

The **mechanical advantage** of the machine is the ratio of the actual load raised to the force required to operate the machine at a constant speed.

$$\text{Mechanical advantage} = \frac{W}{P}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Neglecting any waste of energy in the machine, the work done by P would be equal to the work done in raising the load, and, in these circumstances, the load raised would be larger than W. Let $W_1$ be this hypothetical load, then

$$\text{Work done by } P = \text{work done on } W_1,$$
$$PH = W_1 h,$$
$$W_1 = P\frac{H}{h} = PV. \quad \ldots\ldots\ldots\ldots (3)$$

The effect of frictional and other sources of waste in the actual machine has been to diminish the load from $W_1$ to W. Hence

$$\textbf{Effect of friction} = F = W_1 - W$$
$$= PV - W. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The **efficiency** of any machine is defined as the ratio of the energy delivered to the energy supplied in the same time.

$$\text{Efficiency} = \frac{\text{energy delivered}}{\text{energy supplied}}$$
$$= \frac{Wh}{PH} = \frac{W}{P} \times \frac{1}{V}$$
$$= \frac{\text{mechanical advantage}}{\text{velocity ratio}}. \quad \ldots\ldots\ldots\ldots\ldots (5)$$

**The efficiency thus stated will be always less than unity. Efficiency is often given as a percentage, obtained by multiplying the result given in (5) by 100. 100 per cent. efficiency could be obtained only under**

the condition of no energy being wasted in the machine, a condition impossible to attain in practice.

From equation (3) we have

$$W_1 = P\frac{H}{h},$$

or,
$$\frac{W_1}{P} = \frac{H}{h} = V. \qquad (6)$$

This result shows that the mechanical advantage of an ideal machine, having no waste of energy, is equal to the velocity ratio.

**A typical experiment on a machine.**—In the following experiment a complete record is given of tests on a small crab; this record will serve as a model for any other hoisting machines available.

EXPT. 32.—**Efficiency, etc., of a machine for raising loads.** The machine used was a small crab illustrated in Fig. 201 and shown in outline in Fig. 202. By direct measurement of the distances moved by P and W, the velocity ratio was found to be $V = 8{\cdot}78$. This was confirmed by calculation:

Diameter of barrel to centre of rope sustaining $W = 6{\cdot}4$ inches.

Diameter of wheel to centre of cord sustaining $P = 7{\cdot}9$ inches.

Number of teeth on the barrel wheel, 128.

Number of teeth on the pinion, 18.

Let the barrel make one revolution, then

Height through which W is raised $= \pi \times 6{\cdot}4$ inches.

Number of revolutions of grooved wheel $= \frac{128}{18}$.

Height through which P descends $= \frac{128}{18} \times \pi \times 7{\cdot}9$.

Hence,

$$\text{Velocity ratio} = \frac{128 \times \pi \times 7{\cdot}9}{18 \times \pi \times 6{\cdot}4} = 8{\cdot}78.$$

The weight of the hook from which W was suspended is 1·75 lb. The weight of the scale-pan in which were placed the weights at P is 0·665 lb.

The machine was first oiled, and a series of experiments was made, in each case finding what force P was required to produce **constant speed** in the machine for each value of W. There must be no acceleration, otherwise a portion of P will be utilised in overcoming inertia in the moving parts of the machine, and also in P and W. As the test has for its object the investigation of frictional resistances only, inertia effects must be eliminated, and this is secured by arranging that the speed shall be uniform. The results obtained are given below, together with the calculated values of the loads $W_1$ which could be raised if there were no friction, the effect of friction F, the mechanical advantage and the efficiency.

### Record of Experiments and Results

| (1) W lb. wt., including weight of hook | (2) P lb. wt., including weight of scale-pan | (3) Load $W_1$ if no frictional resistances, $W_1 = PV$ lb. | (4) Effect of friction $F=(W_1-W)$ lb. | (5) Mechanical advantage, $\frac{W}{P}$ | (6) Efficiency, per cent., $\frac{(5)}{V} \times 100$ |
|---|---|---|---|---|---|
| 8·75 | 1·785 | 15·7 | 6·95 | 4·9 | 55·8 |
| 15·75 | 2·665 | 23·4 | 7·65 | 5·9 | 67·2 |
| 22·75 | 3·565 | 31·3 | 8·55 | 6·38 | 72·6 |
| 29·75 | 4·405 | 38·7 | 8·95 | 6·74 | 76·6 |
| 36·75 | 5·335 | 46·8 | 10·05 | 6·89 | 78·5 |
| 43·75 | 6·215 | 54·6 | 10·85 | 7·04 | 80·0 |
| 50·75 | 7·115 | 62·5 | 11·75 | 7·14 | 81·2 |
| 57·75 | 8·065 | 70·8 | 13·05 | 7·16 | 81·6 |
| 64·75 | 8·915 | 78·4 | 13·65 | 7·26 | 82·7 |
| 71·75 | 9·815 | 86·2 | 14·45 | 7·30 | 83·2 |
| 78·75 | 10·705 | 94·1 | 15·35 | 7·36 | 83·7 |
| 85·75 | 11·59 | 101·8 | 16·05 | 7·40 | 84·3 |
| 92·75 | 12·515 | 110 | 17·25 | 7·41 | 84·4 |
| 99·75 | 13·405 | 118 | 18·25 | 7·43 | 84·6 |
| 106·75 | 14·285 | 125·4 | 18·65 | 7·47 | 85·0 |
| 113·75 | 15·205 | 133·8 | 20·05 | 7·48 | 85·2 |
| 120·75 | 16·065 | 141 | 20·25 | 7·51 | 85·5 |
| 127·75 | 16·965 | 149 | 21·25 | 7·53 | 85·7 |

Curves are plotted in Fig. 203 showing the relation of P and W and also that of F and W. It will be noted that these give straight lines. Curves of mechanical advantage and of efficiency in relation to W are

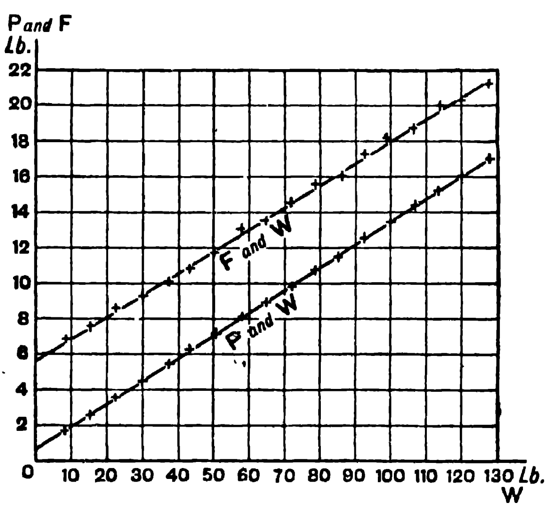



Fig. 203.—Graphs of F and W, and P and W, for a small crab.

shown in Fig. 204. It will be noted that both increase rapidly when the values of W are small and tend to become constant when the value of W is about 120 lb. The efficiency tends to attain a constant value of 86 per cent.

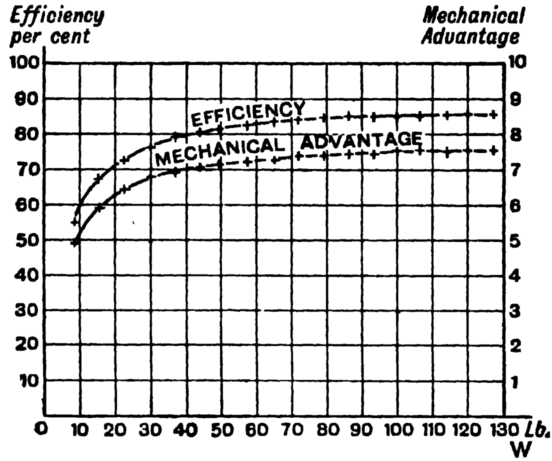



FIG. 204.—Graphs of efficiency and mechanical advantage for a small crab.

As both the curves showing the relation of P and of F with W are straight lines, it follows that the following equations will represent these relations :

$$\mathrm{P}=a\mathrm{W}+b, \qquad (1)$$

$$\mathrm{F}=c\mathrm{W}+d, \qquad (2)$$

where $a$, $b$, $c$ and $d$ are constants to be determined from the graphs.

Select two points on the PW graph, and read the corresponding values of P and W.

P = 3·5 lb. wt. when W = 22·7 lb. wt.

P = 16·0 lb. wt. when W = 120·0 lb. wt.

Hence, from (1),

$$3{\cdot}5=22{\cdot}7a+b,$$

$$16=120a+b.$$

Solving these simultaneous equations, we obtain

$$a=0{\cdot}128, \quad b=0{\cdot}64\,;$$

$$\therefore\ \mathrm{P}=0{\cdot}128\mathrm{W}+0{\cdot}64. \qquad (3)$$

Similarly, When F = 8 lb. wt., W = 20 lb. wt.

When F = 18 lb. wt., W = 100 lb. wt.

Hence, from (2),

$$8=20c+d,$$

$$18=100c+d.$$

The solution of these gives

$$c=0{\cdot}125, \quad d=5{\cdot}5.$$

Hence,

$$\mathrm{F}=0{\cdot}125\mathrm{W}+5{\cdot}5. \qquad (4)$$

If both the load and the hook sustaining the load be removed so that there is no load on the machine, the machine may be **run light**. The values of P and F for this case may be found from (3) and (4) by making W equal to zero, when

$$P = 0{\cdot}64 \text{ lb. wt.}, \quad F = 5{\cdot}5 \text{ lb. wt.}$$

The interpretation is that a force of 0·64 lb. wt. is required to work the machine when running light, and that, if there were no frictional waste, a load of 5·5 lb. wt. could be raised by this force. These values are shown in Fig. 203 by the intercepts on the OY axis between O and the points where the graphs of P and F cut the axis.

**Principle of work applied to levers.**—In Fig. 205 (*a*) is shown a lever AB, pivoted at C, and balanced under the action of two loads W and P. The weight of the lever is neglected. Let the lever be displaced slightly from the horizontal, taking up the position A′CB′. Work has been done by W to the amount of $W \times A'D$, and work has been done on P to the amount of $P \times B'F$. Assuming that there has been no frictional or other waste of energy, we have

$$W \times A'D = P \times B'F.$$

The triangles A′DC and B′FC are similar; hence

$$A'D : B'F = A'C : B'C = AC : BC;$$
$$\therefore \ W \times AC = P \times BC.$$

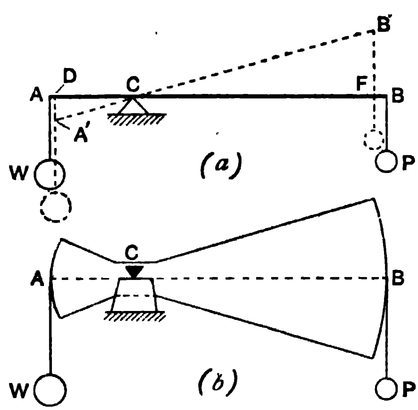


FIG. 205.—Principle of work applied to levers.

This result agrees with that which would have been obtained by application of the principle of moments.

In Fig. 205 (*b*) is shown the same lever with the addition of circular sectors for receiving the cords. It is evident that the arms AC and BC are of constant length in this lever. If the lever is turned through a small angle $\alpha$ radians, W will be lowered through a height $h$ and P will be raised through a height H, and we have

$$\frac{h}{AC} = \alpha = \frac{H}{BC};$$

$$\therefore \ h = AC\,.\,\alpha \quad \text{and} \quad H = BC\,.\,\alpha.$$

Assuming no friction,

$$\text{Work done by } W = \text{work done on } P,$$

$$Wh = PH,$$

or,
$$W \times AC \times \alpha = P \times BC \times \alpha;$$
$$\therefore\ W \times AC = P \times BC,$$
a result which again agrees with the principle of moments.

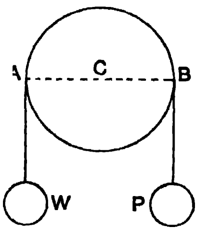


FIG. 206.—Use of a pulley.

In Fig. 206 the sectors are of the same radius and are extended to form a complete wheel. It is evident that P and W will be equal if there be no friction. Such wheels are called **pulleys**, and are much used for changing the direction of a rope or chain under pull, and are found often in tackle used for raising loads.

**Hoisting tackle.**—The fact that the mechanical advantage of a machine, neglecting friction, is equal to the velocity ratio (p. 164) enables the latter to be calculated easily in the following cases of hoisting tackle.

**Simple pulley arrangements.**—In the pulley-block arrangement shown in Fig. 207, let $n$ be the number of ropes leading from the lower to the upper block. Neglecting friction, each rope supports $W/n$; this will also be the value of P. Hence
$$V = \frac{W}{P} = \frac{Wn}{W} = n.$$

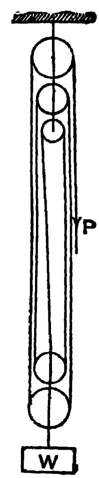


FIG. 207.—A common pulley-block arrangement.

In the arrangement shown in Fig. 208 (seldom used in practice) each rope A and B sustains $\frac{1}{2}W$; the pull in B is balanced by the pulls in C and D, therefore C and D have pulls each equal to $\frac{1}{4}W$; hence E and F have pulls equal to $\frac{1}{8}W$, and the pull in G is also $\frac{1}{8}W$ and is equal to P. Thus
$$V = \frac{W}{P} = \frac{W}{\frac{1}{8}W} = 8.$$

In the arrangement shown in Fig. 208 there are three inverted pulleys. Had there been $n$ inverted pulleys, the value of P would have been
$$P = \frac{W}{2^n} \quad \text{and} \quad V = \frac{W}{P} = 2^n.$$

In the system shown in Fig. 209 (also seldom employed) the pulls in A and B will be each equal to P; hence the pull in C is 2P (neglecting the weight of the pulley), and equals the pull in D. The pull in E is thus 4P and equals the pull in F. Hence
$$W = \text{pull in B} + \text{pull in D} + \text{pull in F} = P + 2P + 4P = 7P$$
$$V = \frac{W}{P} = 7.$$

It is evident that P, 2P and 4P are terms in a geometrical progression having a common ratio 2. Hence, if there be $n$ pulleys, we may write

$$W = P + 2P + 2^2P + 2^3P + \ldots + 2^{n-1}P = P\left(\frac{2^n - 1}{2 - 1}\right) = P(2^n - 1);$$

$$\therefore\ V = \frac{W}{P} = 2^n - 1.$$

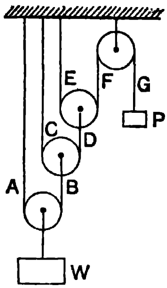



FIG. 208.—Inverted pulley arrangement.

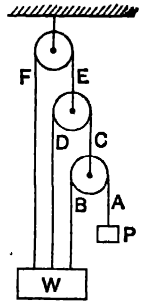



FIG. 209.—Another arrangement of pulleys.

**The Weston's differential blocks** shown in outline in Fig. 210 are much used in practice. The upper block has two pulleys of different diameters, and are fixed together; an endless chain, shown dotted, is arranged as shown. The links of the chain engage with recesses formed in the rims of the pulleys and thus cannot slip. Neglecting friction, each of the chains A and B support $\frac{1}{2}$W. Taking moments about the centre C of the upper pulleys, and calling the radii R and $r$ respectively, we have

$$\tfrac{1}{2}W \times CD = (P \times CF) + (\tfrac{1}{2}W \times CE),$$

$$\tfrac{1}{2}W(R - r) = PR;$$

$$\therefore\ V = \frac{W}{P} = \frac{2R}{R - r}.$$

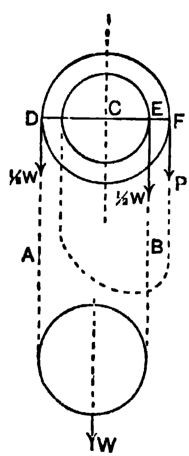



FIG. 210.—Outline diagram of Weston's differential blocks.

Instead of R and $r$, the number of links which can be fitted round the circumferences of the upper pulleys may be used; evidently these will be numbers proportional to R and $r$.

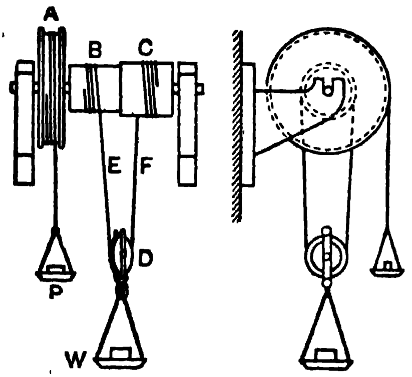
The **wheel and differential axle** (Fig. 211) is a similar contrivance, but has a separate pulley A for receiving the hoisting rope. Taking moments as before, we have

$$PR_A + \tfrac{1}{2}WR_B = \tfrac{1}{2}WR_C,$$

$$PR_A = \tfrac{1}{2}W(R_C - R_B),$$

$$V = \frac{W}{P} = \frac{2R_A}{R_C - R_B}.$$

A set of **helical blocks** is shown in outline in Fig. 212. The pulley A is operated by hand by means of an endless chain, and rotates a **worm** B. The worm is simply a screw cut on the spindle, and engages with the




Front Elevation End Elevation

FIG. 211.—Wheel and differential axle.

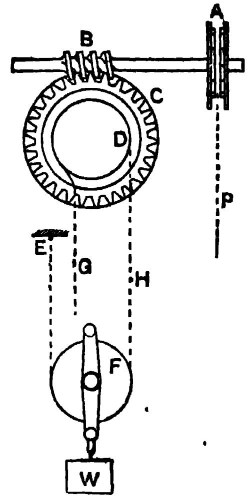



FIG. 212.—Helical blocks.

teeth on a **worm-wheel** C. Each revolution of B causes one tooth on C to advance; hence, if there be $n_C$ teeth on the worm-wheel, B will have to rotate $n_C$ times in order to cause C to make one revolution. Let $L_A$ be the length of the number of links of the operating chain which will pass once round A, then P will advance a distance $n_C L_A$ for one revolution of C. The chain sustaining the load W is fixed at E to the upper block, passes round F, and then is led round D, which has recesses fitting the links in order to prevent slipping. Let $L_D$ be the length of the number of links which will pass once round D, then in one revolution of D, W will be raised a height equal to $\frac{1}{2}L_D$. Hence

$$V = \frac{n_C L_A}{\frac{1}{2}L_D} = \frac{2n_C L_A}{L_D}.$$

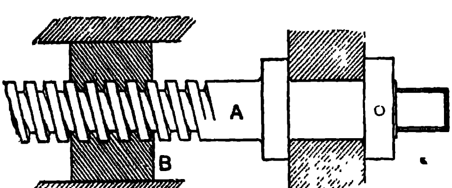



FIG. 213.—Section through a nut B, showing screw, A.

**Screws.**—In Fig. 213, A is a cylindrical piece having a helical groove cut in it, thus leaving a projecting screw-thread which may be of square outline as shown in Fig. 213, or V as in a common bolt. A **helix** may be defined as a curve described on the surface of a cylinder by a point which travels equal distances parallel to the axis of the cylinder for equal angles of rotation. The pitch of the screw is

the distance measured parallel to the axis from a point on one thread to the corresponding point on the next thread. In Fig. 213, B is a sliding block guided so that it cannot rotate, and having a hole with threads to fit those on A. A can rotate, but the collars on it prevent axial movement. One revolution of A will therefore move B through a distance equal to the pitch. If there be $n$ threads per inch, then the pitch $p = 1/n$ inch. The thread shown in Fig. 213 is right-handed; that shown in Fig. 214 is left-handed. Screws are generally made right-handed unless there is some special reason for the contrary; thus the right pedal-pin of a bicycle has generally a left-handed screw where it is fixed to the crank; the action of pedalling then tends to fix it more firmly, whilst a right-handed screw might become unscrewed.

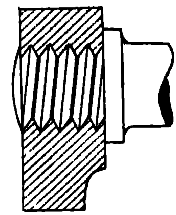


FIG. 214.—A left-handed screw.

In Fig. 215 is shown a differential screw. A has a screw of pitch $p_1$ fitting a screwed hole in B. One revolution of A (the handle moving away from the observer) will advance it towards the left through a distance $p_1$. C has another screw of smaller pitch $p_2$ cut on it, and fits a screwed hole in the sliding block D. If A has no axial movement, D would move towards the right through a distance equal to $p_2$. The actual movement of D towards the left will therefore be $(p_1 - p_2)$ for each revolution of A. By making $p_1$ and $p_2$ very nearly equal, a very slow movement may be given to D.

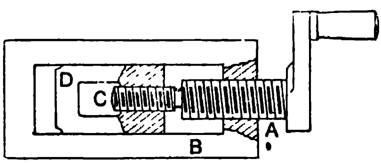



FIG. 215.—A differential screw.

EXPT. 33.—**The screw-jack.** This device for raising loads is shown in Fig. 216. A hollow case A has a hole at the top screwed to receive a square-threaded screw B. The load W lb. weight rests on the top of B; C is a loose collar interposed to prevent the load rotating with the screw. The screw is rotated by means of a bar D. Let a force P lb. weight be applied to D at a distance R inches from the axis of the screw, and let P act horizontally at right angles to the bar. Let the pitch of the screw be $p$ inches. Then, if the screw makes one revolution,

$$\text{Work done by P} = \text{P} \times 2\pi\text{R inch-lb.}$$

$$\text{Work done on W} = \text{W} \times p \text{ inch-lb.}$$

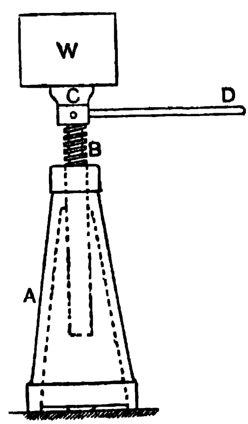



FIG. 216.—Screw-jack.

Assuming that there is no waste of energy, we have

$$W \times p = P \times 2\pi R,$$

or

$$\frac{W}{P} = \frac{2\pi R}{p}.$$

This result gives the mechanical advantage neglecting friction, and is therefore equal to the velocity ratio of the machine. Hence

$$\text{Velocity ratio} = V = \frac{2\pi R}{p}.$$

For experimental purposes the bar D is removed, and a pulley having a grooved rim takes its place. A cord is wound round the rim of the pulley, passes over another fixed pulley and has a scale-pan at its free end. Make a series of experiments with a gradually increasing series of loads W, determining P for each. Reduce the results as directed previously (pp. 164 to 166).

## Exercises on Chapter XIV

**1.** In a set of pulley blocks there are two pulleys in the upper block and one in the lower block. The rope is fastened to the lower block, passes round one of the upper pulleys, then round the lower pulley, and lastly round the other upper pulley. An effort of 70 lb. weight is required to raise a load of 150 lb. weight. Find the velocity ratio and the mechanical advantage, also the effect of friction and the efficiency with this load.

**2.** In a system of pulleys similar to that shown in Fig. 208 there are four movable pulleys each weighing 6 lb. Neglect friction, and calculate what effort must be applied if there is no load. If the efficiency is 60 per cent., reckoned on the work done on W and that done by P, find what effort will be required in order to raise a load of 200 lb. weight.

**3.** A system of pulleys resembling that shown in Fig. 209 has three movable pulleys, each of which weighs 4 lb. Neglecting friction, what effort will be required to sustain a load of 60 lb. weight? If the efficiency is 70 per cent., reckoned as in Question 2, what effort will be required to raise a load of 60 lb. weight?

**4.** The barrel of a crab is 6 inches diameter to the centre of the rope sustaining the load; the wheel on the barrel shaft has 80 teeth, and the pinion gearing with it has 20 teeth. The machine is driven by a handle 15 inches in radius. Find the velocity ratio. If the efficiency is 70 per cent., what load can be raised by an effort of 30 lb. weight applied to the handle? What is the mechanical advantage under these conditions?

**5.** In a Weston's differential pulley block, the numbers of chain links which can be passed round the circumferences of the pulleys are 16 and 15 respectively. Find the velocity ratio of the machine. If a load of 550 lb. weight can be raised by an effort of 20 lb. weight, what are the values of the mechanical advantage, the effect of friction and the efficiency?

**6.** In a wheel and differential axle the wheel is 24 inches in diameter, and the barrel has diameters of 7 and 6 inches respectively. Find the velocity ratio. What load can be raised by an effort of 30 lb. weight if the efficiency is 65 per cent.? Under these conditions, what are the values of the mechanical advantage and the effect of friction?

**7.** In a machine for testing materials under torsion, one end of the test piece is attached to the axle of a worm-wheel and the other end is fixed. The worm-wheel has 90 teeth, and is driven by a worm and hand-wheel. If the hand-wheel is rotated 785 times before the specimen breaks, how many degrees of twist have been given to the specimen? If the average torque on the specimen was 2400 lb.-inches, and if the efficiency of the machine is 70 per cent., how much work was done on the hand-wheel?

**8.** The screw of a screw-jack is 0·5 inch pitch and the handle is 19 inches long. The efficiency is 50 per cent. What effort must be applied to the handle in raising a load of one ton weight? What is the maximum value of the efficiency of any machine?

**9.** A block and tackle is used to raise a load of 200 lb.; the rope passes round three pulleys in the fixed block, and round two in the movable block, to which is fastened the load and one end of the rope. Calculate the force which must be applied to the rope.

Assuming that, owing to the effect of friction, the tension on one side of a pulley is $\frac{4}{5}$ths of the tension on the other side of the pulley, prove that the force required to raise the load must be increased to over 74 lb.

**10.** Define the terms 'velocity ratio', 'mechanical advantage' and 'efficiency' as applied to machines, and show that one of these quantities is equal to the product of the other two. In a lifting machine the velocity ratio is 30 to 1. An effort of 10 lb. is required to raise a load of 35 lb., and an effort of 25 lb. a load of 260 lb. Find the effort required to raise a load of 165 lb. and the efficiency under this load. Assume a linear relation between effort and load.

**11.** How is the work done by a force measured? Define erg, foot-poundal, foot-pound. A vertical rubber cord is stretched by gradually loading a scale-pan attached to its lower end, and a graph is drawn showing the relation between the load and the extension of the cord. Explain how the work done in stretching the cord may be found from the graph. L.U.

**12.** Find the condition of equilibrium for a system of pulleys in which each pulley hangs in the loop of a separate string, the strings being all parallel and each string attached to the beam. The weights of the pulleys are to be taken into account.

If there are 5 pulleys and each weighs 1 lb., what weight will a force equal to the weight of 6 pounds support on such a system, and what will be the total pull on the beam? L.U.

**13.** Find the velocity ratio, mechanical advantage and efficiency of a screw-jack, whose pitch is $\frac{1}{4}$ inch, and the length of whose arm is 15 inches, if the tangential force at the end of the arm necessary to raise one ton is 24 lb. weight.

**14.** Describe the construction of a differential screw, and on the assumption of the principle of work (or otherwise) calculate its velocity ratio. If the two screws have 2 threads and 3 threads to the inch respectively, and a couple of moment 20 lb.-wt.-ft. applied to the differential screw produces a thrust equal to the weight of half a ton, calculate the efficiency of the machine. L.U.

**15.** A body having a weight W is pushed up a rough inclined plane by a force P which acts in a line parallel to the plane. The length, height and base of the plane are L, H and B respectively. Find the work done by P, taking $\mu$ as the coefficient of friction. Show that this work is the same as the work done by a horizontal force in pushing the body along a horizontal plane of length B, and having the same value of $\mu$, and then elevating the body through a height H. Find the mechanical advantage, *i.e.* the ratio W/P, in the case in which $\mu = H/B$.

**16.** Describe the system of pulleys in which the same rope goes round all the pulleys, and find the mechanical advantage (neglecting friction). If one end of the rope is attached to the lower block, and there are five pulleys in all, find the pull which is necessary to raise a mass of one ton. Find also the power required to pull the free end at a speed of 5 ft. per second.

**17.** Find from first principles the velocity ratio in a differential pulley block in which the radii of the wheels are $6\frac{3}{4}$ inches and 7 inches.

With this particular pulley block it is found that pulls of 9 and 33 lb. will just raise weights of 100 and 500 lb. respectively. Assuming that the relation between pull and weight lifted is linear, plot a graph showing how the efficiency varies with the weight raised when the latter is increased up to 2000 lb. J.M.B., H.S.C.

# CHAPTER XV

## MOTION OF ROTATION

**Centre of mass.**—In Fig. 217 is shown a body travelling towards the left in such a manner that every particle has rectilinear motion only; this kind of motion is called **pure translation.** Let the body as a whole have an acceleration $a$, then every particle will have this acceleration. If the masses of the particles be $m_1$, $m_2$, $m_3$, etc., the particles will offer resistances, due to their inertia, given by $m_1a$, $m_2a$, $m_3a$, etc. These forces are parallel; hence the resultant resistance is

$$\mathrm{R} = m_1a + m_2a + m_3a + \text{etc.} = \Sigma ma = a\Sigma m. \quad \text{..................(1)}$$

The centre of these parallel forces (p. 95) is called the **centre of mass** of the body. To find the centre of mass of a thin sheet (Fig. 218), take

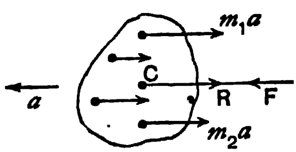



FIG. 217.—Centre of mass of a body.

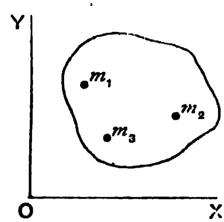



FIG. 218.—Centre of mass of a thin sheet.

reference axes OX and OY. Let the coordinates of $m_1$, $m_2$, $m_3$, etc., be $(x_1y_1)$, $(x_2y_2)$, $(x_3y_3)$, etc. Let the sheet have pure translation parallel to OY, and let the acceleration be $a$. Take moments about O, giving

$$m_1ax_1 + m_2ax_2 + m_3ax_3 + \text{etc.} = a\Sigma mx = a(\Sigma m)\bar{x},$$

where $\bar{x}$ is the abscissa of the centre of mass. Hence

$$\bar{x} = \frac{\Sigma mx}{\Sigma m}. \quad \text{..................................(2)}$$

Similarly, by assuming pure translation with acceleration $a$ parallel to OX, we obtain

$$\bar{y} = \frac{\Sigma my}{\Sigma m}. \quad \text{..................................(3)}$$

The student will note that these equations are similar to those employed for finding the centre of gravity (p. 95), the only difference being the substitution of mass for weight. It may be assumed that the centre of mass coincides with the centre of gravity, and all the methods

employed in Chapter IX may be used for determining the centre of mass. Centre of mass is sometimes called **centroid.**

Referring again to Fig. 217, C is the centre of mass and R is the resultant resistance due to inertia and acts through C. If a force F be applied to the body, and passes through C, it is evident that F and R will act in the same straight line and the motion will be pure translation. The truth of the principle that a force passing through the centre of mass of a body produces no rotation may be tested by laying a pencil on the table and flicking it with the finger nail. An impulse applied near the end of the pencil causes the pencil to fly off, rotating as it goes ; an impulse applied through the centre of mass produces no rotation.

**Rotational inertia.**—To produce pure rotation in a body, *i.e.* the centre of mass remains at rest, requires the application of a couple. The effect of the equal opposing parallel forces is not to produce translational motion. Let a body be free to rotate about an axis OZ perpendicular to the plane of the paper (Fig. 219). Let a couple F, F, be applied, and let the couple rotate with the body so that its effect is constant. The body will have angular acceleration which we proceed to determine.

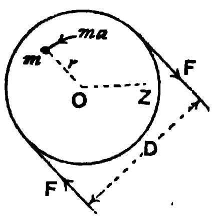



FIG. 219.—Relation between the couple and the rotational inertia.

Consider a particle having a mass $m$ and at a radius $r$ from OZ. Let the particle have a linear acceleration $a$ in the direction of the tangent to its circular path. The inertia of the particle causes it to offer a resistance $ma$. Let $\phi$ be the angular acceleration, then

$$a = \phi r.$$

Also, Resistance of the particle $= ma = m\phi r$.

To obtain the moment of this resistance about OZ, multiply by $r$, giving

$$\text{Moment of resistance of particle} = m\phi r^2 = \phi m r^2.$$

Now $\phi$ is common for the whole of the particles ; hence we have :

$$\text{Total moment of resistance} = \phi(m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \text{etc.})$$
$$= \phi \Sigma m r^2.$$

This moment balances the moment of the applied couple. Let the moment of the couple be $L = FD$, then

$$L = \phi \Sigma m r^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

**It will be noted that L must be stated in absolute units in using this equation.**

$\Sigma mr^2$ is called the second moment of mass, or, more commonly, the **moment of inertia** of the body. It is a quantity which depends upon the mass and the distribution of the mass with reference to the axis of rotation. It is usual to denote it by I, and to add a suffix indicating the axis for which the moment of inertia has been calculated; thus

$$\mathsf{L} = I_{OZ}\phi. \quad \text{..............................(2)}$$

In the C.G.S. system state L in dyne-centimetres, and I in grams mass and centimetre units; in the British system state L in poundal-feet, and I in pounds-mass and foot-units. $\phi$ is in radians per second per second in both systems.

The dimensions of moment of inertia are $ml^2$.

Gravitational units may be employed; thus, if T is the moment of the applied couple in lb.-feet, and I is the moment of inertia in pounds-mass and foot-units, then

$$\mathsf{T} = \frac{I_{OZ}\phi}{g}. \quad \text{..............................(3)}$$

Example 1.—A wheel has a moment of inertia of 800 gram and centimetre units. Find what constant couple must be applied to it in order that the angular acceleration may be 2 radians per second per second.

$$\mathsf{L} = I\phi$$
$$= 800 \times 2 = \underline{1600} \text{ dyne-centimetres.}$$

Example 2.—A grindstone has a moment of inertia of 600 pound and foot units. A constant couple is applied and the grindstone is found to have a speed of 150 revolutions per minute 10 seconds after starting from rest. Find the couple.

$$\omega = \frac{150}{60} \times 2\pi = 5\pi \text{ radians per sec.}$$

$$\phi = \frac{\omega}{t} = \frac{5\pi}{10} = \frac{\pi}{2} \text{ radians per sec. per sec.}$$

$$\mathsf{T} = \frac{I\phi}{g} = \frac{600 \times \pi}{g \times 2}$$
$$= \frac{600 \times 22}{32 \cdot 2 \times 2 \times 7} = \underline{29 \cdot 3} \text{ lb.-feet.}$$

**Cases of moments of inertia.**—A few of the simpler cases of moments of inertia are now discussed.

A thin uniform wire of mass M is arranged parallel to the axis OX (Fig. 220). Every portion of the wire is at the same distance D from the axis; hence

$$I_{OX} = \Sigma m D^2 = MD^2. \quad \text{.................(1)}$$

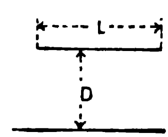


Fig. 220.

The same wire is bent into a circle of radius R (Fig. 221); the axis OZ passes through the centre and is perpendicular to the plane of the circle. Every portion of the wire is at the same distance R from the axis; hence

$$I_{OZ} = \Sigma m R^2 = MR^2. \quad \text{......(2)}$$

A number of such circular wires laid side by side form a tube; hence the moment of inertia of the tube with respect to the longitudinal axis is

$$I_{OZ} = MR^2, \quad \text{......(3)}$$

where M is the total mass of the tube.

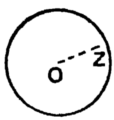



FIG. 221.

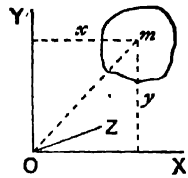



FIG. 222.—$I_{OZ} = I_{OX} + I_{OY}$.

**An important theorem.**—In Fig. 222 is shown a thin plate in the plane of the paper. The coordinates of a small mass $m$, referred to the axis OX, OY are $y$ and $x$. We have for the mass $m$,

$$I_{OX} = my^2; \quad I_{OY} = mx^2;$$

$$\therefore \ I_{OX} + I_{OY} = m(y^2 + x^2) = mr^2,$$

where $r$ is the distance of $m$ from the axis OZ, which passes through O and is perpendicular to the plane of the paper. Since $I_{OZ} = mr^2$, we have for the particle

$$I_{OX} + I_{OY} = I_{OZ}.$$

A similar result can be obtained for any other particle in the plate; hence, for the whole plate,

$$\begin{aligned} I_{OX} + I_{OY} &= m_1(x_1^2 + y_1^2) + m_2(x_2^2 + y_2^2) + m_3(x_3^2 + y_3^2) + \text{etc.} \\ &= m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \text{etc.} \\ &= I_{OZ}. \quad \text{......(4)} \end{aligned}$$

This result enables us to calculate the moment of inertia in cases which would otherwise require mathematical work of some difficulty.

EXAMPLE.—A thin wire of mass M is bent into a circle of radius R (Fig. 223). Find the moment of inertia with respect to a diameter.

Draw the diameters AB and CD intersecting at right angles at O; let OZ be perpendicular to the plane of the circle. Then

$$I_{OZ} = MR^2.$$

Also, $$I_{AB} + I_{CD} = I_{OZ}.$$

From symmetry, $$I_{AB} = I_{CD};$$

$$\therefore \ 2I_{AB} = I_{OZ};$$

$$\therefore \ I_{AB} = \tfrac{1}{2} I_{OZ} = \tfrac{1}{2} MR^2.$$

**Another important theorem.**—In Fig. 224 is shown a thin plate in the plane of the paper. CD is in the same plane and passes through the centre of mass of the plate; OX is also in the same plane and is

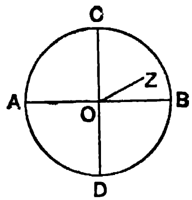


FIG. 223.

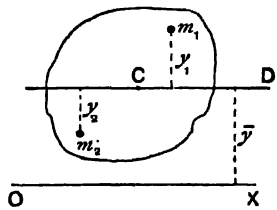


FIG. 224.—$I_{OX} = I_{CD} + M\bar{y}^2$.

parallel to CD, and at a distance $\bar{y}$ from it. To find the relation of $I_{CD}$ and $I_{OX}$, we proceed thus:

Considering the particle $m_1$ at a distance $y_1$ from CD, we have

$$I_{OX} = m_1(y_1 + \bar{y})^2 = m_1 y_1^2 + m_1\bar{y}^2 + 2\bar{y}m_1y_1. \quad \text{.............. (5)}$$

Similarly for $m_2$,

$$I_{OX} = m_2(\bar{y} - y_2)^2 = m_2y_2^2 + m_2\bar{y}^2 - 2\bar{y}m_2y_2. \quad \text{.............. (6)}$$

For all particles above CD the moments of inertia are given by expressions similar to (5), and for particles below CD, by expressions similar to (6); hence the total moment of inertia may be obtained by taking the sum of the equations (5) and (6) for every particle in the plate. The first and second terms in both expressions are similar; the third terms differ only in sign. When all the particles in the plate are considered, the sum of the third terms in (5) and (6) evaluates the product $2\bar{y}$ times the simple moment of mass of the plate about CD. Now CD passes through the centre of mass of the plate, and therefore the simple moment of mass with reference to CD is zero; hence we have for the whole plate

$$I_{OX} = \Sigma my^2 + \Sigma m\bar{y}^2.$$

Since $\bar{y}$ is constant, this reduces to

$$I_{OX} = \Sigma my^2 + \bar{y}^2\Sigma m$$
$$= I_{CD} + M\bar{y}^2, \quad \text{.................................. (7)}$$

where M is the total mass of the plate.

EXAMPLE.—A thin wire of mass M is bent into a circle of radius R. Find the moment of inertia about a tangent.

Let AB (Fig. 225) be a diameter of the circle, and let OX be a tangent parallel to AB. Then

$$I_{AB} = \tfrac{1}{2}MR^2 \text{ (p. 178)}.$$

$$I_{OX} = I_{AB} + MR^2 = \tfrac{1}{2}MR^2 + MR^2 = \tfrac{3}{2}MR^2$$

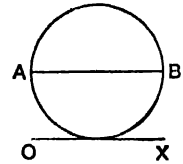


FIG. 225.

***Routh's rule* for calculating the moments of inertia of symmetrical solids.**—If a body is symmetrical about three axes which are mutually perpendicular, the moment of inertia about one axis is equal to the mass of the body multiplied by the sum of the squares of the other two semi-axes and divided by 3, 4 or 5 according as the body is rectangular, elliptical (such as a cylinder), or ellipsoidal (such as a sphere).

EXAMPLE 1.—A rectangular plate (Fig. 226) is symmetrical about GZ and other two axes passing through G and parallel to B and T respectively. Find $I_{GZ}$.

$$I_{GZ}=\frac{M\{(\frac{1}{2}B)^2+(\frac{1}{2}T)^2\}}{3}=\frac{M(B^2+T^2)}{12}. \quad \text{.........(8)}$$

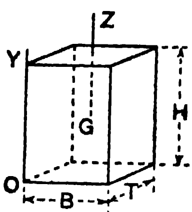



FIG. 226.

EXAMPLE 2.—A solid cylinder (special case of an elliptical body) is symmetrical about the axis OX of the cylinder, and about other two axes forming diameters at 90° and passing through the centre of mass of the cylinder. Find $I_{OX}$.

$$I_{OX}=\frac{M(R^2+R^2)}{4}=\tfrac{1}{2}MR^2. \quad \text{.................(9)}$$

EXAMPLE 3.—A solid sphere (ellipsoidal body) is symmetrical about any three diameters which are mutually perpendicular. Find the moment of inertia with respect to a diameter.

$$I_{OX}=\frac{M(R^2+R^2)}{5}=\tfrac{2}{5}MR^2. \quad \text{..........................(10)}$$

**Other cases of moments of inertia.**—The following results are of service in the solution of problems.

A thin uniform wire, mass M, length L; the axis OX passes through one end and is perpendicular to the wire.

$$I_{OX}=\tfrac{1}{3}ML^2. \quad \text{.................................(11)}$$

A thin rectangular plate, mass M, breadth B, height H; the axis OX coincides with one of the B edges.

$$I_{OX}=\tfrac{1}{3}MH^2. \quad \text{.................................(12)}$$

A thick rectangular plate (Fig. 226); the axis OY coincides with one edge.

$$I_{OY}=\tfrac{1}{3}M(B^2+T^2). \quad \text{.............................(13)}$$

A thin circular plate, mass M, radius R; the axis OZ passes normally through the centre; OX is a diameter.

$$I_{OZ}=\tfrac{1}{2}MR^2. \quad \text{..................................(14)}$$

$$I_{OX}=\tfrac{1}{4}MR^2. \quad \text{..................................(15)}$$

A thin circular plate of mass M having a concentric hole; external

radius $R_1$, internal radius $R_2$; the axis OZ passes normally through the centre.

$$I_{OZ} = \tfrac{1}{2}M(R_1^2 + R_2^2). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

This result also applies to a hollow cylinder having a coaxial hole.

These results may be obtained directly as follows:

Let a uniform rod have length L (Fig. 227) and mass $s$ per unit

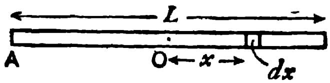



FIG. 227.—Moment of inertia of rod.

length. An element of length $dx$ has mass $s \,.\, dx$ and its moment of inertia about O is $x^2 \,.\, s\,dx$. For the whole rod the moment of inertia is

$$\int_{-\frac{L}{2}}^{+\frac{L}{2}} sx^2 \,.\, dx = \left[\frac{sx^3}{3}\right]_{-\frac{L}{2}}^{+\frac{L}{2}}$$

$$= \frac{2sL^3}{24} = \frac{sL^3}{12}.$$

Now, $sL$ is the mass M of the rod,

$$\therefore \text{ Moment of inertia about O} = \tfrac{1}{12}ML^2.$$

If the centre of rotation is at one end, A, and $x$ is now measured from A,

$$\text{moment of inertia} = \int_0^L sx^2 \,.\, dx$$

$$= \tfrac{1}{3}ML^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

For the rectangular solid of Fig. 226, consider the plan shown in Fig. 228. A slice of thickness $dx$ has volume TH . $dx$ and mass $\rho$TH . $dx$, where $\rho$ is the density of the material. The moment of inertia about an axis A is therefore $\frac{1}{12}\rho TH \,.\, T^2 \,.\, dx$, as above. The moment of inertia about O is $\frac{1}{12}\rho T^3H \,.\, dx + \rho THx^2dx$ (p. 179), and for the whole solid,

$$\text{Moment of inertia} = \tfrac{1}{12}\rho T^3H\int_{-\frac{B}{2}}^{+\frac{B}{2}} dx + \rho TH\int_{-\frac{B}{2}}^{+\frac{B}{2}} x^2\,dx$$

$$= \tfrac{1}{12}\rho T^3H \,.\, B + \rho TH\left[\frac{x^3}{3}\right]_{-\frac{B}{2}}^{+\frac{B}{2}}$$

$$= \tfrac{1}{12}\rho T^3HB + \tfrac{1}{12}\rho THB^3$$

$$= \tfrac{1}{12}\rho THB(B^2 + T^2)$$

$$= \tfrac{1}{12}M(B^2 + T^2),$$

where M is the mass $\rho$THB.

If the moment of inertia about C (Fig. 228) is required, note that

$$I_C = I_D + M \cdot OC^2$$
$$= \tfrac{1}{12}M(B^2+T^2) + M\left\{\left(\frac{B}{2}\right)^2 + \left(\frac{T}{2}\right)^2\right\}$$
$$= \tfrac{1}{3}M(B^2+T^2). \quad \ldots\ldots (13)$$

A cylinder or disc is seen in plan in Fig. 229. A cylindrical slice of radius $r$ and thickness $dr$ has volume $2\pi r \cdot dr \cdot L$, where L is the length

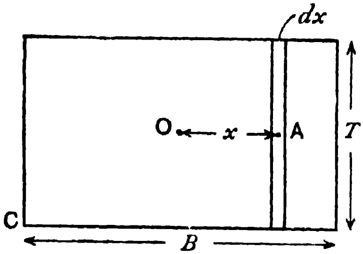




FIG. 228.—Moment of inertia of rectangular solid.

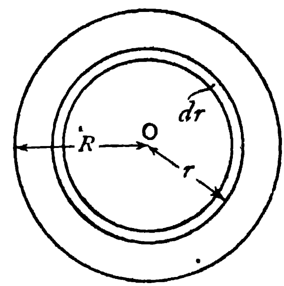




FIG. 229.—Moment of inertia of cylinder.

of the cylinder, and mass $2\pi\rho r L \cdot dr$, where $\rho$ is its density. The moment of inertia of this is $2\pi\rho L r dr \times r^2 = 2\pi\rho L r^3 dr$. For the whole solid cylinder the moment of inertia is

$$2\pi\rho L\int_0^R r^3\,dr = 2\pi\rho L \cdot \frac{R^4}{4}.$$

But the mass M of the cylinder is $\pi R^2 L\rho$;

$$\therefore \text{ Moment of inertia about } O = \tfrac{1}{2}MR^2. \quad \ldots\ldots (14)$$

**Radius of gyration.**—The radius of gyration of a body with respect to a given axis is defined as a quantity $k$ such that, if its square be multiplied by the mass of the body, the result gives the moment of inertia of the body with reference to that axis. Thus

$$I = Mk^2,$$
$$k = \sqrt{\frac{I}{M}}.$$

For example, a solid cylinder $I = \frac{MR^2}{2}$ with respect to the axis of the cylinder; hence

$$k = \sqrt{\frac{\frac{1}{2}MR^2}{M}} = \sqrt{\frac{R^2}{2}} = \frac{R}{\sqrt{2}}.$$

EXAMPLE.—In a laboratory experiment a flywheel of mass 100 pounds and radius of gyration 1·25 feet (Fig. 230) is mounted so that it may be rotated by a falling weight attached to a cord wrapped round the wheel

axle. Neglecting friction, find what will be the accelerations if a body of 10 lb. weight is attached to the cord; the radius of the axle is 2 inches.

Let

$M$ = the mass attached to the cord, in pounds.

$Mg$ = its weight, in poundals.

$T$ = pull in the cord, in poundals.

$r$ = radius of the axle, in feet.

$I$ = moment of inertia of wheel
$= 100 \times 1{\cdot}25 \times 1{\cdot}25 = 156{\cdot}2$ pound and foot units.

$a$ = the linear acceleration of $M$, in feet per sec. per sec.

$\phi$ = the angular acceleration of the wheel, in radians per sec. per sec.

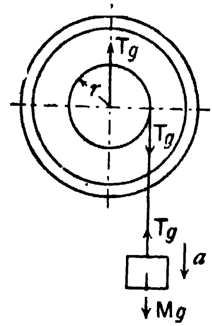



FIG. 230.— An experimental flywheel.

Then, considering $M$, we have

$$Mg - T = Ma. \quad \text{.................................(1)}$$

Considering the wheel, we have

$$Tr = I\phi. \quad \text{.................................(2)}$$

Also,

$$\phi = \frac{a}{r}. \quad \text{(p. 48).} \quad \text{.........................(3)}$$

These three equations enable the solution to be obtained. Thus:

From (2) and (3),

$$Tr = I\frac{a}{r};$$

$$\therefore\ T = I\frac{a}{r^2}. \quad \text{.................................(4)}$$

Substituting in (1) gives

$$Mg - I\frac{a}{r^2} = Ma,$$

$$Mg = \left(M + \frac{I}{r^2}\right)a;$$

$$\therefore\ a = \frac{Mg}{M + \frac{I}{r^2}} = \frac{10 \times 32{\cdot}2}{10 + (156{\cdot}2 \times 6 \times 6)}$$

$$= \underline{0{\cdot}0572} \text{ feet per sec. per sec.}$$

From (3),

$$\phi = \frac{0{\cdot}0572}{r} = 0{\cdot}0572 \times 6$$

$$= \underline{0{\cdot}343} \text{ radian per sec. per sec.}$$

**Angular momentum.**—The **angular momentum** or **moment of momentum** of a particle may be explained by reference to Fig. 231. A particle of mass $m$ revolves in the circumference of a circle of radius $r$ and has a linear velocity $v$ at any instant in the direction of the tangent. Hence

its linear momentum at any instant is given by $mv$. Now $v$ is equal to $\omega r$, when $\omega$ is the angular velocity; hence

$$\text{Linear momentum of the particle} = \omega mr. \quad\text{.................}(1)$$

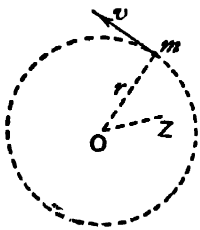


FIG. 231.—Angular momentum of a body.

The moment of this momentum about OZ (Fig. 231) may be obtained by multiplying by $r$, the result being called the **moment of momentum, or angular momentum.**

$$\text{Angular momentum of the particle} = \omega mr^2. \quad\text{.....}(2)$$

Each particle in a body rotating about OZ would have its angular momentum given by an expression similar to (2); hence

$$\text{Angular momentum of a body} = \omega \Sigma mr^2 = \omega I_{OZ}. \quad\text{..................}(3)$$

Consider now a body free to rotate about a fixed axis, and, starting from rest, to be acted upon by a constant couple L. The constant angular acceleration being $\phi$, we have

$$L = I_{OZ}\phi \text{ (p. 177).}$$

Let L act during a time $t$ seconds, then the angular velocity $\omega$ at the end of this time will be

$$\omega = \phi t, \quad \text{or} \quad \phi = \frac{\omega}{t}. \quad \text{(p. 48)}$$

Hence,

$$L = \frac{\omega I_{OZ}}{t}. \quad\text{..................................}(4)$$

Now $\omega I_{OZ}$ is the angular momentum acquired in the time $t$ seconds; hence $\omega I_{OZ}/t$ will be the change in angular momentum per second. We may therefore write

$$L = \text{change in angular momentum per second.} \quad\text{............}(5)$$

If the couple is expressed in gravitational units, say T, we have

$$T = \frac{\omega I_{OZ}}{gt}. \quad\text{.................................}(5')$$

If the angular velocity of a rotating body be changed from $\omega_1$ to $\omega_2$ in $t$ seconds, then

$$\phi = \frac{\omega_1 - \omega_2}{t}. \quad \text{(p. 47)}$$

and the couple required is given by

$$L = \left(\frac{\omega_1 - \omega_2}{t}\right) I_{OZ}, \text{ absolute units,} \quad\text{.....................}(6)$$

or

$$T = \left(\frac{\omega_1 - \omega_2}{gt}\right) I_{OZ}, \text{ gravitational units.} \quad\text{.............}(6')$$

**Kinetic energy of a rotating body.**—In Fig. 232 is shown a body rotating with uniform angular velocity $\omega$ about an axis OZ perpendicular to the plane of the paper. Consider the particle $m_1$, having a linear velocity $v_1$.

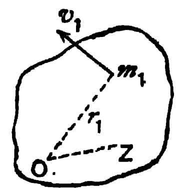



FIG. 232.—Kinetic energy of rotation.

Kinetic energy of the particle $= \frac{m_1 v_1^2}{2}$. (p. 146.)

Now, $$v_1 = \omega r_1;$$

$$\therefore v_1^2 = \omega^2 r_1^2.$$

Hence

$$\text{Kinetic energy of particle} = \frac{m\omega^2 r_1^2}{2}$$

$$= \frac{\omega^2}{2} \cdot m r_1^2. \quad \ldots\ldots\ldots\ldots (1)$$

A similar expression would result for any other particle; hence

$$\text{Total kinetic energy of the body} = \frac{\omega^2}{2} \Sigma m r^2$$

$$= \frac{\omega^2}{2} \mathrm{I}_{OZ} \text{ absolute units } \ldots\ldots\ldots (2)$$

$$= \frac{\omega^2}{2g} \mathrm{I}_{OZ} \text{ gravitational units. } \ldots. (2')$$

EXAMPLE 1.—A wheel has a mass of 5000 pounds and a radius of gyration of 4 feet. Find its kinetic energy at 150 revolutions per minute.

$$\omega = \tfrac{150}{60} \times 2\pi = 5\pi \text{ radians per sec.}$$

$$\mathrm{I} = \mathrm{M}k^2 = 5000 \times 4 \times 4 = 80{,}000 \text{ pound and foot units.}$$

$$\text{Kinetic energy} = \frac{\omega^2}{2g}\mathrm{I} = \frac{25 \times \pi^2 \times 80{,}000}{64 \cdot 4}$$

$$= \underline{306{,}500} \text{ foot-lb.}$$

EXAMPLE 2.—The above wheel slows from 150 to 148 revolutions per minute. Find the energy which has been abstracted.

$$\text{Change in kinetic energy} = \frac{\omega_1^2}{2g}\mathrm{I} - \frac{\omega_2^2}{2g}\mathrm{I}$$

$$= (\omega_1^2 - \omega_2^2)\frac{\mathrm{I}}{2g}.$$

Also, $$\omega_1 = 5\pi,$$

$$\omega_2 = \tfrac{148}{60} \cdot 2\pi = 4 \cdot 933\pi;$$

$$\therefore \text{Energy abstracted} = (\omega_1 - \omega_2)(\omega_1 + \omega_2)\frac{\mathrm{I}}{2g}$$

$$= 0 \cdot 067 \times 9 \cdot 933 \times \frac{80{,}000 \times \pi^2}{64 \cdot 4}$$

$$= \underline{8{,}160} \text{ foot-lb.}$$

**Energy of a rolling wheel.**—The total kinetic energy of a wheel rolling with uniform speed along a road may be separated into two parts, viz. the kinetic energy due to the motion of translation, and the kinetic energy due to the motion of rotation. The total kinetic energy will be the sum of these.

Let, $\omega$ = the angular velocity.

$v$ = the linear velocity of the carriage to which the wheel is attached (this will also be the velocity of the centre of the wheel).

$\mathsf{M}$ = the mass of the wheel.

$k$ = its radius of gyration with reference to the axle.

Then, Kinetic energy of rotation $= \frac{\omega^2 I}{2} = \frac{\omega^2 \mathsf{M} k^2}{2}$.

Kinetic energy of translation $= \frac{\mathsf{M} v^2}{2}$.

$$\text{Total kinetic energy} = \frac{\omega^2 \mathsf{M} k^2}{2} + \frac{\mathsf{M} v^2}{2}. \quad \text{............(1)}$$

Further, if there be no slipping between the wheel and the road, *i.e.* perfect rolling, we have

$$\omega = \frac{v}{\mathsf{R}}, \quad \text{...................................(2)}$$

where $\mathsf{R}$ is the radius of the wheel.

Substituting in (1), we obtain for perfect rolling :

$$\text{Total kinetic energy} = \frac{v^2 \mathsf{M} k^2}{2\mathsf{R}^2} + \frac{\mathsf{M} v^2}{2}$$

$$= \frac{\mathsf{M} v^2}{2}\left(\frac{k^2}{\mathsf{R}^2} + 1\right) \text{ absolute units} \quad \text{............(3)}$$

$$= \frac{\mathsf{M} v^2}{2g}\left(\frac{k^2}{\mathsf{R}^2} + 1\right) \text{ gravitational units.} \quad \text{......(3')}$$

**Energy of a wheel rolling down an inclined plane.**—Fig. 233 illustrates the case of a wheel rolling from $\mathsf{A}$ to $\mathsf{B}$ down an inclined plane. $\mathsf{A}$ is at a height $\mathsf{H}$ above $\mathsf{B}$. Assuming that no energy is wasted, we may apply the principle of the conservation of energy.

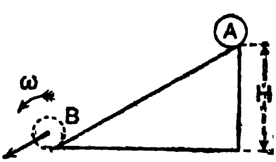



FIG. 233.—Energy of a wheel rolling down an incline.

Potential energy at $\mathsf{A}$

$= \mathsf{M}g\mathsf{H}$

$=$ total kinetic energy at $\mathsf{B}$.

Let $\mathsf{M}$ and $\mathsf{R}$ be respectively the mass and

radius of the wheel, and let $v$ and $\omega$ be the linear and angular velocities at B. As there is supposed to be no waste of energy, there will be no slipping which would lead to waste in overcoming frictional resistances. Hence

$$v = \omega \mathrm{R}.$$

Using equation (3) (p. 186) for the total kinetic energy, we obtain

$$\mathrm{M}g\mathrm{H} = \frac{\mathrm{M}v^2}{2}\left(\frac{k^2}{\mathrm{R}^2}+1\right),$$

or,

$$v = \sqrt{\frac{2g\mathrm{H}}{\frac{k^2}{\mathrm{R}^2}+1}}. \quad \text{.............................(1)}$$

**Motion of a wheel rolling down an incline.**—The above problem may be studied in the following manner. In Fig. 234 a wheel is rolling without slipping down a plane inclined at an angle $\alpha$ to the horizontal. Resolve the weight M$g$ into components M$g \sin \alpha$ and M$g \cos \alpha$ respectively parallel and at right angles to the incline. The normal reaction Q of the incline will be equal to M$g \cos \alpha$, since no acceleration takes place in the direction perpendicular to the incline. The force of friction, F, acts on the wheel tangentially in the direction of the incline.

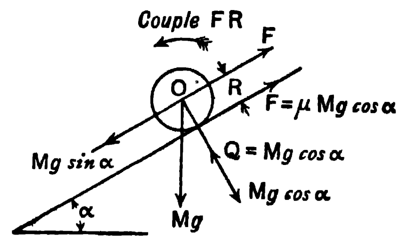



Fig. 234.—Motion of a wheel rolling down an incline.

The effect of F may be ascertained by transferring it to the centre of mass O of the wheel (p. 115), introducing at the same time a couple of anticlockwise moment FR. The wheel is now under the action of opposing forces M$g \sin \alpha$ and F, both applied at O in a direction parallel to the incline, together with a couple FR. The forces produce a linear acceleration $a$ given by

$$\mathrm{M}g \sin \alpha - \mathrm{F} = \mathrm{M}a. \quad \text{.............................(1)}$$

The couple produces an angular acceleration $\phi$ given by

$$\phi = \frac{\mathrm{FR}}{\mathrm{I_{OZ}}} = \frac{\mathrm{FR}}{\mathrm{M}k^2}. \quad \text{.............................(2)}$$

Also, since there is no slipping,

$$\phi = \frac{a}{\mathrm{R}}. \quad \text{.............................(3)}$$

From (2),

$$\mathrm{F} = \frac{\phi \mathrm{M}k^2}{\mathrm{R}}.$$

Substituting in (1), we obtain

$$\mathrm{M}g \sin \alpha - \frac{\phi \mathrm{M}k^2}{\mathrm{R}} = \mathrm{M}a.$$

Substituting for $\phi$ from (3), gives

$$g \sin \alpha - \frac{ak^2}{R^2} = a,$$

$$a\left(1 + \frac{k^2}{R^2}\right) = g \sin \alpha,$$

$$a = \frac{g \sin \alpha}{1 + \frac{k^2}{R^2}}. \quad \text{......................(4)}$$

Suppose that the body starts from rest at A (Fig. 235) and rolls to B. The linear velocity of the centre of mass when at B may be calculated thus:

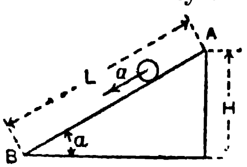



FIG. 235.

$$v^2 = 2aL \quad \text{(p. 26)}.$$

Also,

$$\frac{H}{L} = \sin \alpha; \quad \text{or,} \quad L = \frac{H}{\sin \alpha};$$

$$\therefore \; v^2 = 2a \frac{H}{\sin \alpha}.$$

Inserting the value of $a$ from (4), we have

$$v^2 = \frac{2g \sin \alpha}{1 + \frac{k^2}{R^2}} \cdot \frac{H}{\sin \alpha} = \frac{2gH}{1 + \frac{k^2}{R^2}};$$

$$\therefore \; v = \sqrt{\frac{2gH}{1 + \frac{k^2}{R^2}}}. \quad \text{......................................(5)}$$

Comparison of this with equation (1) (p. 187) indicates that the results obtained by both methods agree.

EXPT. 34.—**Kinetic energy of a flywheel.** In this experiment the wheel is driven by means of a falling weight attached to a cord which is wrapped round the wheel axle and looped to a peg on the axle so that the cord disengages when unwound (Fig. 236).

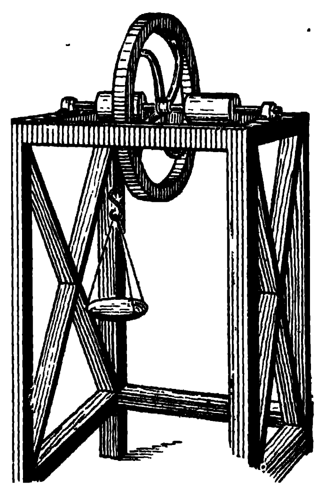


FIG. 236.—Experimental flywheel.

Weigh the scale-pan and let its mass together with that of the load placed in it be M. Let the scale-pan touch the floor and let the cord be taut; turn the wheel by hand through $n_1$ revolutions (a chalk mark on the rim helps), and measure the height H through which the scale-pan is elevated. Allow the scale-pan to descend, being careful not to assist the wheel to

start; note the time of descent; repeat three or four times and take the average time $t$ seconds. Again allow the scale-pan to descend three or four times, and note the total revolutions of the wheel from starting to stopping, being careful not to interfere with it in any way; let the average revolutions be $n_2$. Repeat the experiment, using different values of M and of H. Tabulate the results:

EXPERIMENT ON A FLYWHEEL

| No. of Expt. | Load M | Height H | Time of descent $t$ secs. | Revs. while M is descending $n_1$ | Total revs. $n_2$ |
|---|---|---|---|---|---|
| | | | | | |

Reduce the results for each experiment as follows:

**Energy available** = potential energy given up by M

$$= MgH. \qquad (1)$$

**Disposal of energy:**

(*a*) The energy of M at the instant it reaches the floor is given by

$$\text{Kinetic energy acquired by } M = \frac{Mv^2}{2}.$$

To find the velocity $v$ at the instant M reaches the floor, we have

$$\text{Average velocity} \times t = H;$$

$$\therefore \text{ Average velocity} = \frac{H}{t}.$$

$$\text{Final velocity} = v = \frac{2H}{t};$$

$$\therefore \text{ Kinetic energy acquired by } M = K_M = \frac{M}{2} \cdot \frac{4H^2}{t^2} = \frac{2MH^2}{t^2}. \qquad (2)$$

Hence the energy which has been given to the wheel up to the instant that M reaches the floor is, from (1) and (2),

$$MgH - \frac{2MH^2}{t^2} = MgH - K_M. \qquad (3)$$

(*b*) Some of this energy has been wasted in overcoming the friction of the bearings. Ultimately the whole of the energy given by (3) is so wasted during the entire period of motion, *i.e.* in $n_2$ revolutions of the wheel. Hence,

$$\text{Energy wasted per revolution} = \frac{MgH - K_M}{n_2},$$

and energy wasted whilst M is descending

$$= \left(\frac{MgH - K_M}{n_2}\right) n_1. \qquad (4)$$

(c) The kinetic energy possessed by the wheel, at the instant M reaches the floor, may be calculated by deducting the kinetic energy acquired by M, together with the energy wasted in overcoming friction whilst M is descending, from the energy available. Let this energy be K, then

$$K = MgH - K_M - \left(\frac{MgH - K_M}{n_2}\right)n_1. \quad \text{.......................(5)}$$

During the descent of M, the wheel has made $n_1$ revolutions in $t$ seconds. Let N be the maximum speed in revolutions per second, then

$$\text{Average speed} \times t = n_1\,;$$

$$\therefore\ \text{average speed} = \frac{n_1}{t},$$

and,

$$N = \frac{2n_1}{t}. \quad \text{......................................(6)}$$

The kinetic energy is proportional to the square of the speed, hence Kinetic energy of the wheel at 1 revolution per second

$$= \frac{K}{N^2}. \quad \text{......................................(7)}$$

Obtain the value of this for each experiment; there should be fair agreement. Take the average result and call it $K_1$. Then

$$K_1 = \frac{\omega^2}{2}I, \quad \text{(p. 185)}$$

and,

$$\omega = 2\pi \text{ radians per sec.};$$

$$\therefore\ K_1 = \frac{4\pi^2}{2}I = 2\pi^2 I\,;$$

$$\therefore\ I = \frac{K_1}{2\pi^2}. \quad \text{......................................(8)}$$

The final result gives the moment of inertia of the wheel.

EXPT. 35.—**A wheel rolling down an incline.** A convenient form of apparatus is shown in Fig. 237; the incline consists of two bars AB and the axle

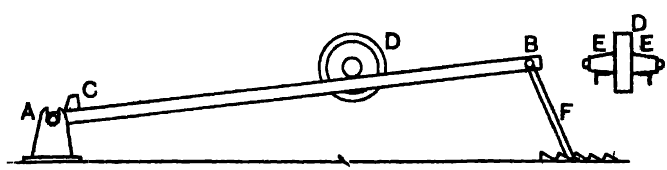



FIG. 237.—Apparatus for investigating the motion of a wheel rolling down an incline.

E E of the wheel D rolls on them. The time of descent is thus increased, and it becomes possible to measure it with fair accuracy by means of a stop-watch.

Set the incline to a suitable inclination by means of the adjustable prop F. Measure the height from the horizontal table to the centre of the

wheel axle; first, when the wheel is at the top, and second, when the wheel is at the bottom of the incline. Let the difference in the heights be H. Measure the diameter of the wheel axle, and hence find its radius R. Allow the wheel to roll down, being careful not to assist it to start; note the time of descent. Repeat several times, and take the average time $t$ seconds. Measure the length of the incline traversed by the wheel; let this be L.

Then, $$\text{Average velocity} \times t = \mathrm{L};$$

$$\therefore \text{ average velocity} = \frac{\mathrm{L}}{t},$$

and, $$\text{maximum velocity} = v = \frac{2\mathrm{L}}{t}. \quad \text{.........................(1)}$$

Evaluate this velocity and substitute in equation (1), p. 187, giving

$$v^2 = \frac{4\mathrm{L}^2}{t^2} = \frac{2g\mathrm{H}}{\frac{k^2}{\mathrm{R}^2} + 1},$$

whence, $$k^2 = \left(\frac{2g\mathrm{H}t^2}{4\mathrm{L}^2} - 1\right)\mathrm{R}^2. \quad \text{............................(2)}$$

Weigh the wheel with its axle in order to determine its mass M Then

$$\text{Moment of inertia of the wheel} = \mathrm{M}k^2$$

$$= \left(\frac{2g\mathrm{H}t^2}{4\mathrm{L}^2} - 1\right)\mathrm{MR}^2. \quad \text{..........(3)}$$

Repeat the experiment, giving different slopes to the incline, and calculate the moment of inertia for each experiment; take the average value.

## Exercises on Chapter XV

1. A wheel has a moment of inertia of 10,000 in pound and foot units. If the wheel starts from rest and acquires a speed of 200 revolutions per minute in 25 seconds, what constant couple has been acting on it?

2. A wheel has a mass of 6 kilograms, and its radius of gyration is 20 centimetres. If its speed be changed from 8000 to 7000 revolutions per minute in 10 seconds, what constant couple has opposed the motion?

3. A wheel is acted upon by a constant couple of 650 poundal-inches; starting from rest it makes 6 revolutions in the first 8 seconds. What is the moment of inertia of the wheel?

4. A thin straight rod 6 feet long has a mass of 0·4 pound. Find its moment of inertia with respect to (*a*) an axis parallel to the rod and 8 inches from it; (*b*) an axis perpendicular to the rod and passing through one end; (*c*) an axis perpendicular to the rod and passing through its centre.

5. The rod given in Question 4 is bent into a complete circle. Find its moment of inertia with respect to (*a*) an axis passing through the centre of the circle and perpendicular to its plane; (*b*) a diameter of the circle; (*c*) a tangent.

6. A thin circular plate has a mass of 2 pounds and the radius is 9 inches. Find the moment of inertia with respect to the following axes: (*a*) passing through the centre and perpendicular to the plane of the plate:

(*b*) a diameter; (*c*) a tangent; (*d*) a line perpendicular to the plane of the plate and passing through a point on the circumference; (*e*) a similar line to that given in (*d*), but bisecting a radius.

**7.** A thin rectangular plate has a mass of 1·5 pounds; the edges are 3 feet and 2 feet respectively. Find the moment of inertia with respect to (*a*) a 3 feet edge; (*b*) a 2 feet edge; (*c*) a line parallel to the 3 feet edges and bisecting the plate; (*d*) a line parallel to the 2 feet edges and bisecting the plate; (*e*) a line perpendicular to the plane of the plate and passing through the intersection of the diagonals; (*f*) a line perpendicular to the plane of the plate and passing through one corner.

**8.** An iron plate, 4 feet high, 2 feet wide and 2 inches thick, is hinged at a vertical edge. Find the moment of inertia with respect to the axis of the hinges. The density of iron is 480 pounds per cubic foot.

**9.** A hollow cylinder of iron is 60 feet long, 20 inches external and 8 inches internal diameter. The density is 480 pounds per cubic foot. Find the moment of inertia about the axis of the cylinder.

**10.** A solid sphere of cast iron is 12 inches in diameter. The density is 450 pounds per cubic foot. Find the moment of inertia about a diameter, and also about a tangent.

**11.** A wheel having a mass of 50 tons and a radius of gyration of 15 feet runs at 50 revolutions per minute. It is observed to take 4·5 minutes in coming to rest. What steady couple has been acting?

**12.** A wheel is mounted in bearings so that the axis of rotation is horizontal, and is driven by a cord wrapped round the axle and carrying a load. The axle is 4 inches diameter measured to the centre of the cord. A preliminary experiment shows that a load of 2 lb. weight produces steady rotation, the wheel being assisted to start by hand. The load is then increased to 4 lb. weight; starting from rest, this load descended 3 feet in 6·5 seconds. Find the moment of inertia of the wheel.

**13.** A solid disc, 3 feet in diameter, has a mass of 200 pounds. Calculate its angular momentum when rotating 300 times per minute. If the speed is changed to 320 revolutions per minute in 40 seconds, what constant couple has been applied?

**14.** A thin iron rod, 2 feet long, mass 0·6 pound, revolves about an axis perpendicular to and bisecting the rod. If the speed is 120 revolutions per minute, find the moment of momentum. If a couple of 0·3 lb.-feet be applied for 2 seconds so as to increase the speed, find the final speed of rotation.

**15.** Calculate the kinetic energy of a wheel having a moment of inertia of 30,000 in pound and foot units, when rotating 180 times per minute. How much energy does the wheel give up in changing speed to 179 revolutions per minute?

**16.** A bicycle wheel, 28 inches in diameter, has a mass of 2 pounds, and the radius of gyration is 13 inches. The bicycle is travelling at 12 miles per hour. Find (*a*) the kinetic energy of rotation of the wheel; (*b*) its kinetic energy of translation; (*c*) its total kinetic energy.

**17.** A solid cylinder, mass 4 pounds, diameter 6 inches, starts from rest at the top and rolls without slipping down a plane inclined at 5° to the horizontal. If the incline is 10 feet long, find the kinetic energies of translation and rotation when the cylinder reaches the bottom.

**18.** Find the linear and angular accelerations of the cylinder given in Question 17.

**19.** Two cylinders, A and B, have the same over-all dimensions and their masses are equal. The cylinder A has a lead core and the outer part is wood; the cylinder B has a wooden core and the outer part is lead. Both cylinders start simultaneously from rest at the top of an incline and roll without slipping. Which cylinder will reach the bottom first? Give reasons for your answer.

**20.** Write down expressions for the coordinates of the centre of mass of a number of particles of given mass, the coordinates of whose positions are given.

A uniform square plate of 1 ft. side has two circular holes punched in it, one of radius 1 inch, coordinates of centre (4, 5) inches, referred to two adjacent sides of the plate as axes, the other of radius $\frac{1}{2}$ inch, coordinates of centre (8, 1) inches; find the coordinates of the centre of mass of the remainder of the plate.

**21.** Write down an expression for the kinetic energy of a wheel whose moment of inertia is I, rotating $n$ times a second.

A wheel has a cord of length 10 feet coiled round its axle; the cord is pulled with a constant force of 25 lb. wt., and when the cord leaves the axle, the wheel is rotating 5 times a second. Calculate the moment of inertia of the wheel. L.U.

**22.** A hollow circular cylinder, of mass M, can rotate freely about an external generator (*i.e.* a straight line drawn on the curved surface and parallel to the axis of the cylinder), which is horizontal. Its cross-section consists of concentric circles of radii 3 and 5 feet. Show that its moment of inertia about the fixed generator is 42M units, and find the least angular velocity with which the cylinder must be started when it is in equilibrium, so that it may just make a complete revolution. L.U.

**23.** A projectile whose radius of gyration about its axis is 5 inches is fired from a rifled gun, and on leaving the gun its total kinetic energy is 50 times as great as its kinetic energy of rotation. How far does the projectile travel on leaving the gun before making one complete turn? L.U.

**24.** On what does the inertia of a body, with respect to rotation about an axis, depend?

Prove that the energy of rotation of a small mass whirled in a circle is equal to half the product of its rotation-inertia (moment of inertia) about the axis of rotation into the square of its angular velocity.

Show that one-half of the kinetic energy acquired by a hoop in rolling down an inclined plane is rotational.

**25.** What is meant by 'moment of inertia' of a body? Show that the moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through its centre of mass, plus the moment of inertia which the body would have about the given axis if all collected at its centre of mass.

**26.** A wheel runs at 240 revolutions per minute, and is required to give up 10,000 foot-lb. of energy without the speed falling below 239 revolutions per minute. Calculate the moment of inertia which the wheel must have. If the radius of gyration is 5 feet, find the mass of the wheel.

**27.** Two flywheels in the same plane are free to rotate about horizontal parallel frictionless axles. A light cord has one end attached to the first

wheel, is wound round it, passes over to the second wheel, is wound round that, and has its other end attached to the second wheel. A couple of moment G acts on the first wheel so as to make the cord start coiling up on the first wheel and uncoiling from the second. Prove that the angular acceleration of the second wheel is $Gab/(Ib^2+Ja^2)$, where $a$ and $b$ are the radii of the wheels, and I and J are their moments of inertia about their respective axles. Find also the tension of the cord. J.M.B., H.S.C.

**28.** A uniform circular disc of mass $m$ and radius $a$ rolls along a straight line on a horizontal plane with its own plane vertical, being pulled forward by a horizontal force P applied at its centre. Prove that the frictional force brought into play at the point of contact is $Pk^2/(a^2+k^2)$, where $k$ is the radius of gyration of the disc about its centre.

If $\mu$ is the coefficient of limiting friction between the disc and the plane and if $P>\mu mg(1+a^2/k^2)$, prove that the disc will slip on the plane, and that the forward acceleration of the point of contact will be $(P/m)-\mu g(1+a^2/k^2)$.

J.M.B., H.S.C.

**29.** A light string hangs over a pulley which is a uniform circular disc of radius 1·8 cm. and mass 400 gm. At the ends of the string there hang masses of 200 and 250 gm. and the system starts from rest. If the string does not slip on the pulley and the axle-friction of the pulley is negligible, find (*a*) the kinetic energy of the system when the 250 mass has descended 1 cm., (*b*) the angular acceleration of the pulley. J.M.B., H.S.C.

# CHAPTER XVI

## CENTRIFUGAL FORCE. PENDULUMS

**Centrifugal force.**—It has been shown (p. 38) that when a particle moves in the circumference of a circle of radius R with uniform velocity $v$ (Fig. 238) there is a constant acceleration towards the centre of the circle given by

$$a = \frac{v^2}{R}.$$

To produce this acceleration requires the application of a uniform force F, also continually directed towards the centre of the circle and given by

$$F = ma = \frac{mv^2}{R} \text{ absolute units, } \dots\dots\dots(1)$$

or, $$P = \frac{mv^2}{gR} \text{ gravitational units. } \dots\dots\dots(1')$$

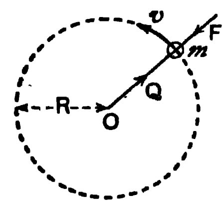


FIG. 238.—Central and centrifugal forces.

The force F overcomes the inertia of the particle, which would otherwise pursue a straight line path, and may be called the **central force** (sometimes called the **centripetal force**). It is resisted by an equal opposite force Q (Fig. 238) due to the inertia of the particle. Q is called the **centrifugal force.**

Expressed in terms of the angular velocity,

$$F = Q = \frac{m\omega^2 R^2}{R} = \omega^2 mR. \dots\dots\dots\dots\dots\dots\dots\dots\dots(2)$$

Since $mR$ is the simple moment of mass of the particle with reference to the axis of rotation, it follows that in a large body, consisting of many particles, the centrifugal force may be calculated by imagining the whole mass of the body to be concentrated at the centre of mass. Let M be the mass of the body and let Y be the radius drawn to the centre of mass from the axis of rotation (Fig. 239), then

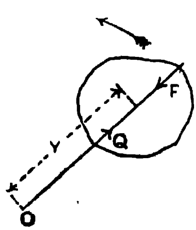


FIG. 239.—Resultant cenetrifugal force.

$$\text{Centrifugal force} = \omega^2 MY \text{ absolute units. } \dots\dots\dots(3)$$

$$= \frac{\omega^2}{g} MY \text{ gravitational units. } \dots(3')$$

It follows from this result that if a body rotates about an axis passing through its centre of mass (in which case $Y=0$), there will be no resultant force on the axis due to centrifugal action. If the body is not symmetrical, a disturbing couple may act on the axis. Thus in Fig. 240 is shown a rod rotating about an axis GX, G being the centre of

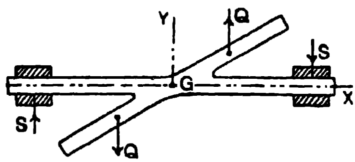



FIG. 240.—Rocking couple due to want of symmetry.

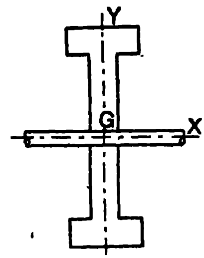



FIG. 241.—A balanced symmetrical body.

mass. The rod is not symmetrical about GY; hence, considering the halves separately, there will be centrifugal forces Q, Q forming a couple tending to bring the rod into the axis GY. To balance this tendency, the bearings must apply forces S, S, forming a couple equal and opposite to that produced by Q, Q. These forces will, of course, rotate with the rod and produce what is called a **rocking couple**. In Fig. 241 is shown a body symmetrical about GY, and consequently having neither rocking couple nor resultant centrifugal force; in other words, this body is completely balanced.

**Centrifugal force on vehicles.**—In Fig. 242 is shown the front view of a motor-car moving in a path curved in plan. To prevent side-slipping, the road is banked up to such an extent that the resultant Q of the centrifugal force and the weight falls perpendicularly to the road surface.

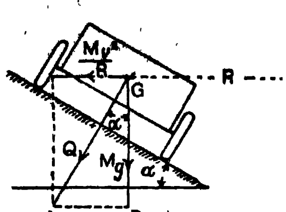



FIG. 242.—Section of a banked motor track.

Let

$M$ = the mass of the car.

$v$ = the velocity.

$R$ = the radius of the curve, as seen in the plan.

Then, $\qquad$ Centrifugal force $=\frac{Mv^2}{R}$ absolute units.

Weight of car $=Mg$ absolute units.

The triangle of forces is ABG; hence

$$\frac{\text{Centrifugal force}}{\text{Weight of car}}=\frac{Mv^2}{RMg}=\frac{v^2}{gR}=\frac{AB}{BG}.$$

Now, $$\frac{AB}{BG} = \tan\alpha \text{ (Fig. 242)},$$

and $\alpha$ is also the angle which the section of the road surface makes with the horizontal; hence

$$\tan\alpha = \frac{v^2}{gR}.$$

Railway tracks are banked in a similar manner; the outer rail is laid at a greater elevation than that on the inside of the curve, and so grinding of the flanges of the outer wheels against the rail is prevented.

A cyclist turning a corner instinctively leans inwards (Fig. 243). The forces acting on machine and rider are the total weight $Mg$, the centrifugal force $Mv^2/R$—where R is the radius of the curve and $v$ is the velocity, the vertical reaction of the ground Q—equal to $Mg$, and a frictional force F applied to the wheels by the ground. If all goes well, the clockwise couple formed by $Mg$ and Q is balanced by the anticlockwise couple formed by F and $Mv^2/R$. It is evident that the higher the speed and the smaller the radius of the curve, the greater will be the centrifugal force, and the rider will have to lean inwards at a greater angle. Since the centrifugal force and the friction are equal, it may happen that the limiting value of the friction may be attained, when a side-slip will ensue.

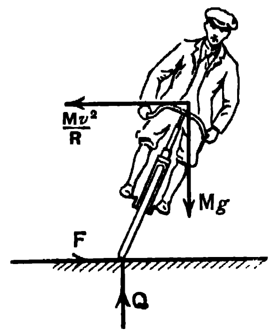


FIG. 243.—A cyclist turning a corner.

**Simple harmonic motion.**—In Fig. 244, AB is any diameter of the circle and NS is another diameter at right angles to AB. Let a point P travel in the circumference of the circle with constant velocity $v$; draw PM perpendicular to NS. It will be seen that M, the projection of P on NS, vibrates in NS as P moves round the circumference of the circle. The motion of M is called **simple harmonic motion, or vibration.**

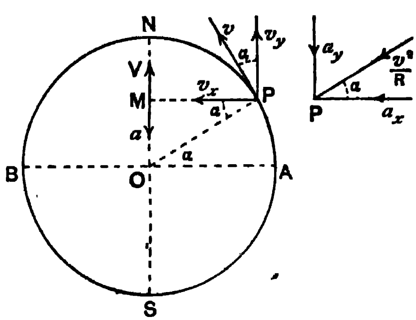


FIG. 244.—Simple harmonic motion.

Let the radius of the circle be R, and let the angle $\alpha$ described by OP be measured from the initial position OA. The angular velocity $\omega$ of OP is $v/R$, and the displacement of M from the middle of the vibration at any instant is given by

$$OM = OP \sin\alpha = R \sin\alpha. \quad \text{......................(1)}$$

Let $t$ seconds be the time in which OP describes the angle $\alpha$, then $\alpha = \omega t$, and we may write

$$\text{OM} = \text{R} \sin \omega t. \quad \text{.............................} (1')$$

Denoting as positive the displacements above O, and as negative those below O, the algebraic sign of $\sin \omega t$ will determine on which side of O the point M falls at the end of the time $t$. The maximum displacement ON or OS is called the **amplitude** of the vibration.

**Velocity and acceleration in simple harmonic motion.**—To obtain the velocity V of M, take components $v_x$ and $v_y$ of the velocity of P respectively parallel and perpendicular to AB (Fig. 244). Since $v_y$ is perpendicular to OA and $v$ is perpendicular to OP, it follows that the angle between $v$ and $v_y$ is equal to $\alpha$. Hence

$$v_y = v \cos \alpha = v \cos \omega t.$$

The component $v_x$ does not affect the velocity of M, therefore

$$\text{V} = v \cos \alpha. \quad \text{..................................} (2)$$

$$= v \cos \omega t. \quad \text{..................................} (2')$$

$$= \omega \text{R} \cos \omega t. \quad \text{..............................} (2'')$$

To obtain the acceleration $a$ of M, resolve the central acceleration of P, viz. $v^2/\text{R}$, into components $a_x$ and $a_y$ respectively parallel and perpendicular to AB, as shown separately in Fig. 244. The component $a_x$ does not affect the motion of M ; hence

$$a = a_y = \frac{v^2}{\text{R}} \sin \alpha. \quad \text{..............................} (3)$$

$$= \frac{v^2}{\text{R}} \sin \omega t. \quad \text{..........................} (3')$$

$$= \omega^2 \text{R} \sin \omega t. \quad \text{.........................} (3'')$$

It will be noticed from (2) that the velocity of M is proportional to $\cos \alpha$. Now $\cos \alpha = \text{PM}/\text{OP}$ and is therefore proportional to PM ; hence V is proportional to PM. V is zero when M is at N and also when M is at S. Maximum value of V occurs when M is passing through O and is given by

$$\text{V}_{\text{max.}} = v \cos 0 = v. \quad \text{..........................} (4)$$

The algebraic sign of $\cos \alpha$ indicates whether V is from S towards N (positive), or from N towards S (negative). From (1′) and (3″) we may write for the acceleration of M,

$$a = \omega^2 \times \text{displacement of M from O}. \quad \text{.................} (5)$$

Hence the acceleration of M is proportional to the displacement from the middle of the vibration. The algebraic sign of $\sin \alpha$ in (3) indicates whether $a$ is from N towards O (positive), or from S towards O (negative) (Fig. 244). It will be noted that the acceleration is directed constantly towards O. From (3), $a$ has zero value when $\sin \alpha = 0$, *i.e.*

when $\alpha=0$ or $\pi$; M will then be passing through O. Maximum values of $a$ occur when $\sin\alpha=\pm 1$, *i.e.* when M is at N and again at S; in these positions

$$a_{\text{max.}}=\pm\frac{v^2}{R}=\pm\omega^2 R. \qquad \text{.............................(6)}$$

Displacement, velocity and acceleration diagrams for M have been drawn in Figs. 245 (*a*), (*b*), and (*c*) for values of $\alpha$ from 0 to $2\pi$. It is evident that the displacement and acceleration graphs are curves of sines, and that the velocity graph is a curve of cosines. Further, since $\alpha$ is proportional to $t$, it follows that these diagrams are also displacement, velocity and acceleration graphs on time bases; the base line O to $2\pi$ represents the time of one revolution of P (Fig. 244), or the time of one complete vibration of M from N to S and back to N. This time is called the **period** of the vibration.

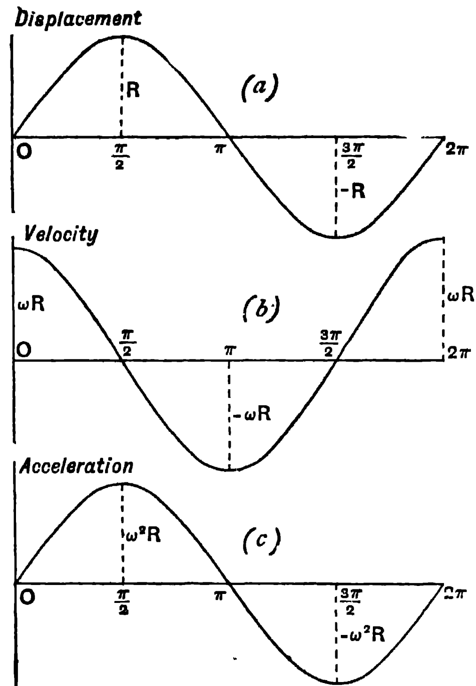


FIG. 245.—Graphs for simple harmonic motion.

Let $T=$ the period, then

$$vT=2\pi R \text{ (Fig. 244)},$$

$$T=\frac{2\pi R}{v}, \qquad \text{...........(7)}$$

$$=\frac{2\pi R}{\omega R}=\frac{2\pi}{\omega}. \qquad \text{...(7')}$$

The **frequency** of the vibration is the number of vibrations per second, and is obtained by taking the reciprocal of T; thus

$$\text{Frequency}=n=\frac{1}{T}, \text{ vibrations per sec. .......} \qquad \text{.(8)}$$

EXAMPLE.—A point describes simple harmonic vibrations in a line 4 cm. long. Its velocity when passing through the centre of the line is 12 cm. per second. Find the period.

The given maximum value of V is also the velocity of P in the circumference of the circle (Fig. 244); hence

$$T=\frac{2\pi R}{v}=\frac{2\times 22\times 2}{12\times 7}=\underline{1{\cdot}05} \text{ seconds.}$$

A well-known mechanism in which simple harmonic motion is realised is shown in Fig. 246. A crank revolves in the dotted circle about a fixed centre and engages a block which may slide in a slotted bar. Rods attached to the bar are guided so as to be capable of vertical motion only. The effect of the slot is to cancel the horizontal components of the velocity and acceleration of the crank pin; hence the vertical motion of the rods is simple harmonic.

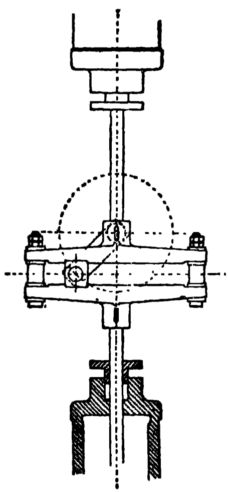


FIG. 246.—Slotted-bar mechanism.

**Forces required in simple harmonic vibrations.**—Referring to Fig. 247, in which a particle of mass $m$ is executing simple harmonic vibrations in NS, the force F required to overcome inertia when the particle is at C is given by

$$F = ma.$$

Substituting the value of $a$ obtained in (3″), p. 198, we have

$$F = m\omega^2 R \sin \omega t$$
$$= m\omega^2 \times \text{displacement OC (Fig. 247)}. \quad \ldots\ldots(1)$$

Hence F is proportional to the displacement OC and is directed constantly towards the middle of the vibration. The maximum values of F occur when the particle is at N and at S, and are given by

$$F_{max.} = \pm m\omega^2 \times ON. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Let $\mu$ be the value of F when the particle is at unit displacement from O, then

$$\mu = m\omega^2, \quad \text{or,} \quad \omega^2 = \frac{\mu}{m}.$$

The period of the vibration is given by (7′), p. 199.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{\mu}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

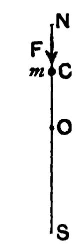



FIG. 247.—Force required in simple harmonic motion.

In using this result $\mu$ must be stated in absolute units.

EXAMPLE.—A body of mass 2 grams executes simple harmonic vibrations. When at a distance of 3 cm. from the centre of the vibration, a force of 0·4 gram weight is acting on it. Find the period.

$$\mu = \frac{0{\cdot}4}{3} = 0{\cdot}133 \text{ gram weight} = 0{\cdot}133 \times 981 = 130{\cdot}3 \text{ dynes}.$$

$$T = 2\pi\sqrt{\frac{m}{\mu}} = \frac{2 \times 22}{7}\sqrt{\frac{2}{130{\cdot}3}} = \underline{0{\cdot}777} \text{ second}.$$

**The simple pendulum.**—A simple pendulum may be realised by attaching a small body to a light thread and allowing it to execute small vibrations in a vertical plane under the action of gravity (Fig. 248). The forces acting on the small body at B are its weight, $mg$, and the pull T of the thread. The resultant of these is a force F, which may be taken as horizontal if the angle BAD is kept very small, and may be obtained from the triangle of forces ABD.

$$\frac{\mathrm{F}}{mg}=\frac{\mathrm{BD}}{\mathrm{AD}}, \quad \text{or,} \quad \mathrm{F}=mg\frac{\mathrm{BD}}{\mathrm{AD}}.$$

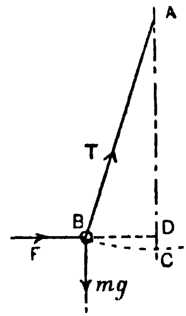



FIG. 248.—A simple pendulum.

Now, if the angle BAD is very small, AC and AD will be sensibly equal. Let L be the length of the thread, then,

$$\mathrm{F}=mg\frac{\mathrm{BD}}{\mathrm{AC}}=\frac{mg}{\mathrm{L}}.\mathrm{BD}. \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Hence we may say that F is proportional to BD for vibrations of small amplitude. BD and BC coincide nearly for such vibrations, and the body will execute simple harmonic vibrations under the action of a force F which varies as the displacement of B from the vertical through A. To obtain the value of $\mu$, make BD = 1 in (1), giving

$$\mu=\frac{mg}{\mathrm{L}}.$$

Now,

$$\mathrm{T}=2\pi\sqrt{\frac{m}{\mu}}=2\pi\sqrt{\frac{m\mathrm{L}}{mg}}$$

$$=2\pi\sqrt{\frac{\mathrm{L}}{g}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

EXAMPLE.—Find the period of a simple pendulum of length 4 feet at a place when $g$ is 32 feet per second per second. Find also the frequency.

$$\mathrm{T}=\frac{2\times 22}{7}\sqrt{\frac{4}{32}}=\underline{2{\cdot}222} \text{ seconds.}$$

$$n=\frac{1}{\mathrm{T}}=\frac{1}{2{\cdot}222}=\underline{0{\cdot}449} \text{ vibration per second.}$$

**Vibrations differing in phase.**—In Fig. 249 ($a$), two points $P_1$ and $P_2$ rotate in the circumference of the circle with equal and constant angular velocities. Their projections $M_1$ and $M_2$ on AB execute simple harmonic vibrations which are said to differ in **phase**. The p[illegible] difference may be defined as the value of the constant angle $P_1OP_2 =$ [illegible] and may be stated in degrees or radians. Thus a phase difference of 90° or $\pi/2$ radians would give vibrations, such that $M_1$ would be at the end

A of the vibration, at the instant that $M_2$ was passing through the point O.

The vibrations possessed separately by $M_1$ and $M_2$ may be impressed on a single particle, which will then execute simple harmonic vibra-

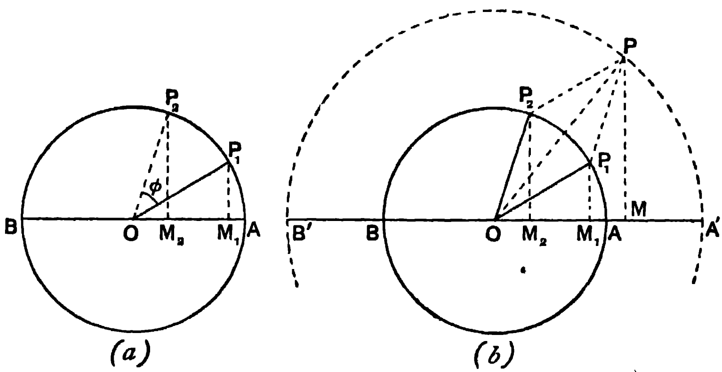



FIG. 249.—Vibrations differing in phase.

tions compounded of the vibrations possessed by $M_1$ and $M_2$. In Fig. 249 (*b*) construct a parallelogram by making $P_1P$ and $P_2P$ equal and parallel respectively to $OP_2$ and $OP_1$. Join OP and draw PM perpendicular to BA produced.

$OM_2$ and $M_1M$ are equal, since they are the projections on AB of equal lines equally inclined to AB. Therefore OM is equal to the sum of the component displacements $OM_1$ and $OM_2$. Hence, if the parallelogram $OP_1PP_2$ rotate about O with the same angular velocity possessed originally by $OP_1$ and $OP_2$, then M will execute simple harmonic vibrations in A′B′, and will have a resultant motion of which the vibrations of $M_1$ and $M_2$ are the components.

It will be evident now that if two simple harmonic vibrations in the same straight line, of equal amplitudes and periods but differing in phase by 180°, be impressed on the same particle, the particle will remain at rest.

For further examples of simple harmonic vibrations, the student is referred to Part IV of the volume devoted to Sound.

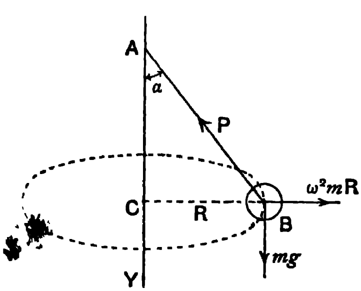



FIG. 250.—A conical pendulum.

**Conical pendulum.**—The conical pendulum consists of a small heavy particle B (Fig. 250) attached to a light cord AB which is attached at the upper end to a fixed point A. The axis AY is vertical and the particle B describes a circular path, the plane of which is horizontal. AB in revolving generates a conical surface.

Let $m$ = the mass of the particle,
$\omega$ = the constant angular velocity, radians/sec.,
$\alpha$ = the angle between AB and AY,
$\mathrm{R}$ = the radius of the circle described by B,
$\mathrm{H}$ = the height AC of the cone

The forces acting on the particle are its weight $mg$, the centrifugal force $\omega^2 m\mathrm{R}$, and the tension P of the cord. These forces balance, and the triangle of forces for them is ABC. Hence

$$\frac{\mathrm{AC}}{\mathrm{CB}}=\frac{\mathrm{H}}{\mathrm{R}}=\frac{mg}{\omega^2 m\mathrm{R}}=\frac{g}{\omega^2\mathrm{R}};$$

$$\therefore\ \mathrm{H}=\frac{g\mathrm{R}}{\omega^2\mathrm{R}}=\frac{g}{\omega^2}. \quad \text{.................................(1)}$$

It will be seen from this result that the height of the cone is independent of the mass of the particle and of the length of the cord AB; it depends solely on the angular velocity and on the value of $g$. For a given value of H at a stated place, for which the value of $g$ is known, $\omega$ has a definite value, and hence the time of one revolution has a fixed value.

Let T = the time in sec of 1 revolution.

Then, $\omega\mathrm{T}=2\pi$, or, $\mathrm{T}=2\pi/\omega$.

From (1), $\omega=\sqrt{\frac{g}{\mathrm{H}}}$;

$$\therefore\ \mathrm{T}=2\pi\sqrt{\frac{\mathrm{H}}{g}}. \quad \text{..................................................(2)}$$

Referring again to Fig. 250, we have from (1),

$$\cos\alpha=\frac{\mathrm{H}}{\mathrm{AB}}=\frac{g}{\omega^2\mathrm{AB}}. \quad \text{.............................(3)}$$

Also,

$$\frac{\mathrm{P}}{mg}=\frac{\mathrm{AB}}{\mathrm{AC}}=\frac{\mathrm{AB}}{\mathrm{H}}=\frac{\mathrm{AB}\,.\,\omega^2}{g};$$

$$\therefore\ \mathrm{P}=m\omega^2\mathrm{AB}. \quad \text{.................................(4)}$$

If the angular velocity be changed from $\omega_1$ to $\omega_2$, there will be a corresponding alteration in the height of the cone from $\mathrm{H}_1$ to $\mathrm{H}_2$. Thus, from (1),

$$\mathrm{H}_1=\frac{g}{\omega_1^2}, \quad \text{and} \quad \mathrm{H}_2=\frac{g}{\omega_2^2}$$

$$\therefore\ \mathrm{H}_1-\mathrm{H}_2=g\left(\frac{1}{\omega_1^2}-\frac{1}{\omega_2^2}\right). \quad \text{.................................(5)}$$

If $\omega_2$ be greater than $\omega_1$, $(\mathrm{H}_1-\mathrm{H}_2)$ is a decrease in the height of the

cone, *i.e.* the particle rises ; if $\omega_2$ be less than $\omega_1$, the particle takes up a lower position.

This fact renders the conical pendulum useful as an engine governor, an example of which is shown in Fig. 251. The vertical spindle is driven by the engine and has two arms pivoted near the top. These arms carry masses which realise the ideal particle in the conical governor. Other arms connect the masses to a sleeve which can slide on the spindle. Movements of the sleeve as the speed changes are communicated by the bent lever and rod to a throttle valve in the steam pipe. Increase in speed causes the masses to rise ; hence the sleeve also rises and the movement partially closes the throttle valve, thus reducing the quantity of steam passing to the engine and hence reducing the speed. Reduction in speed is followed by an inverse action, and more steam is supplied to the engine.

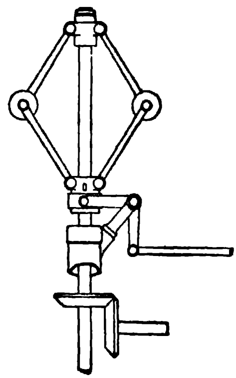


FIG. 251.—An engine governor.

**Loaded governor or conical pendulum.**—The speed of revolution of the simple governor shown in Fig. 251 is limited to about 70 or 80 revolutions per minute ; at higher speeds H becomes too small to be suitable for fulfilling the function of governing. The speed may be increased and these defects avoided by the device of loading the sleeve (Fig. 252 (*a*)). In this governor the sleeve carries a mass of M units ; all four arms are inclined at the same angle $\alpha$ to the vertical. Each of the pins $C_1$ and $C_2$ sustains $\frac{1}{2}Mg$ ; also $C_1$ is balanced under the action of three forces, $\frac{1}{2}Mg$, the pull P in $C_1A_1$, and a horizontal force Q supplied by the sleeve (Fig. 252 (*b*)). The pull P is transmitted by the link $A_1C_1$, and applies a force P to the mass $A_1$ ; this force may be resolved into a vertical force $\frac{1}{2}Mg$ and a horizontal force Q. From the triangle of forces $A_1D_1C_1$, we have

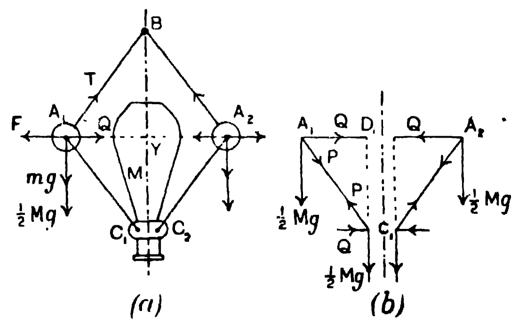



FIG. 252.—Loaded Porter governor.

$$\frac{Q}{\frac{1}{2}Mg} = \frac{A_1D_1}{D_1C_1} = \tan A_1C_1D_1,$$

or,

$$Q = \tfrac{1}{2}Mg \tan \alpha. \qquad \text{.....................................(1)}$$

Referring to Fig. 252 (*a*), we see that $A_1$ is subjected to three forces. viz. the pull T in the upper arm $A_1B$, the resultant $(F - Q)$ of the centri-

fugal force F and Q, and the resultant $(mg+\frac{1}{2}Mg)$ of the weights. $A_1BY$ is the triangle of forces, and we have

$$\frac{F-Q}{mg+\frac{1}{2}Mg}=\frac{YA_1}{BY}=\tan\alpha. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Let the angular velocity be $\omega$, and let $A_1Y=R$ and $BY=H$, then, from (1) and (2),

$$\frac{\omega^2mR-\frac{1}{2}Mg\frac{R}{H}}{mg+\frac{1}{2}Mg}=\frac{R}{H};$$

$$\therefore\ \omega^2mH-\frac{1}{2}Mg=mg+\frac{1}{2}Mg;$$

$$\therefore\ \omega^2=\left(\frac{M+m}{m}\right)\frac{g}{H}$$

$$=\left(\frac{M}{m}+1\right)\frac{g}{H}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$H=\left(\frac{M}{m}+1\right)\frac{g}{\omega^2}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

These results show that H is increased by an addition to the mass M; by adjusting M suitably, the governor arms can be made to run at the most desirable angle to the vertical, whatever may be the normal speed of revolution.

EXPT. 36.—**Determination of the value of $g$ by means of a simple pendulum.** Arrange a simple pendulum by attaching a small heavy bob to one end of a long silk cord. Take a series of readings with varying lengths of cord, in each case taking care that the angle of vibration is small. For each length of cord L, note the time of 100 complete vibrations, and hence determine the period of vibration, T seconds.

$$T=2\pi\sqrt{\frac{L}{g}};$$

$$\therefore\ T^2\propto L.$$

Plot the values of $T^2$ and L obtained in the experiment; the resulting graph should be a straight line. From the graph determine the average value of the ratio

$$r=\frac{L}{T^2};$$

then

$$g=4\pi^2\times r.$$

EXPT. 37.—**Longitudinal vibrations of a helical spring.** Hang a helical spring from a rigid support and attach a load to the lower end. Apply an additional smaller load and measure the extension produced by it. From the result calculate the force required to give unit extension to the spring. Remove the additional load; gently pull the load downwards and let go. Since the extension of the spring is proportional to the pull applied (p. 133), the force at any instant tending to return the load to

the initial position is proportional to the displacement from this position. Hence the load will have simple harmonic vibrations. The spring also vibrates, and may be taken into account by adding one-third of its mass to that of the load.

Let $m$ = the mass of the load + $\frac{1}{3}$ mass of the spring.

$\mu$ = the force required to produce unit extension of the spring.

Then, $\mathsf{T}=2\pi\sqrt{\frac{m}{\mu}}$ seconds (p. 200).

Evaluate this time, and check it by finding the period of vibration experimentally. Do this by finding the time $t$ taken to execute 100 vibrations, when

$$\mathsf{T}=\frac{t}{100}.$$

Let $m_1$ be the mass of the load required to give unit extension to the spring, then $\mu=m_1g$, and $\mathsf{T}=2\pi\sqrt{m/m_1g}$, therefore $g=4\pi^2m/\mathsf{T}^2m_1$. Hence calculate the value of $g$ from the experimental quantities.

**Torsional oscillations.**—In Fig. 253 is shown a vertical wire AB, firmly clamped at A to a fixed support and also at B to a body C. When at rest, the body occupies a position which may be called the position of static equilibrium. If C be rotated slightly about the vertical axis against the torsional resistance of the wire and then released, the body will execute **torsional oscillations,** and will describe angles on each side of the position of static equilibrium. Now when a twisting torque is applied to a wire, within the limits of Hooke's law (p. 129), we know that the angle through which one end of the wire twists relative to the other end is proportional to the torque (p. 134). Hence when the body C is oscillating under the constraint of the wire, the couple tending to restore it to the position of static equilibrium is proportional to the angular displacement from this position. This condition is analogous to that of a body executing simple harmonic vibrations under the influence of a restoring force which is proportional to the displacement from the centre of the vibration. Hence an expression for the period of torsional oscillations may be obtained from the equation (3) on p. 200 by writing the restoring couple $\lambda$ absolute units at unit displacement (one radian) instead of the force $\mu$, and the moment of inertia I of the oscillating body instead of the mass $m$. Hence

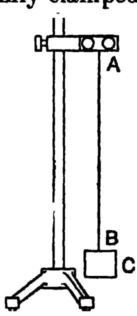


FIG. 253.—Torsional oscillations.

$$\mathsf{T}=2\pi\sqrt{\frac{\mathrm{I}}{\lambda}}\text{ seconds.} \quad \text{.............................(1)}$$

EXAMPLE 1.—It is found that a couple of 0·3 lb.-ft. is required to twist a certain wire through 90°. If a solid cylinder 4 inches in diameter and

having a mass of 8 pounds is attached coaxially to the wire, find the period of torsional oscillation.

$$I=\frac{MR^2}{2} \text{ (p. 182)}$$
$$=\frac{8\times 2^2}{2\times 12^2}=\frac{1}{9} \text{ pound and foot units.}$$
$$\text{Couple}=0\cdot 3\times 32\cdot 2=9\cdot 66 \text{ poundal-feet for } 90°.$$
$$\lambda=9\cdot 66\times\frac{57\cdot 3}{90}=6\cdot 15 \text{ poundal-feet for 1 radian.}$$
$$T=2\pi\sqrt{\frac{I}{\lambda}}$$
$$=2\pi\sqrt{\frac{1}{9\times 6\cdot 15}}=\frac{2}{3}\pi\sqrt{\frac{1}{6\cdot 15}}$$
$$=\frac{2}{3}\pi\times\frac{1}{2\cdot 48}=\underline{0\cdot 846} \text{ sec.}$$

EXAMPLE 2.—A thin cylindrical tube has a diameter of 10 cm. and the mass is 240 grams. When executing torsional oscillations under the influence of a wire the period is found to be 2 seconds. Find the couple required to twist the wire through one complete revolution.

$$I=MR^2$$
$$=240\times 5\times 5=6000 \text{ gram and cm. units.}$$
$$T=2\pi\sqrt{\frac{I}{\lambda}};$$
$$\therefore\ T^2=4\pi^2\frac{I}{\lambda},$$
$$\lambda=\frac{4\pi^2 I}{T^2}=4\pi^2\times\frac{6000}{4}$$
$$=59{,}400 \text{ dyne-cm. per radian.}$$
$$\therefore\ \text{Couple for 1 rev.}=59{,}400\times 2\pi$$
$$=\underline{373{,}300} \text{ dyne-cm.}$$

If the oscillating body is of a simple character, *e.g.* a solid cylinder soldered to the end of the wire, the moment of inertia can be calculated easily. The apparatus illustrated in Fig. 254 avoids any difficult calculation. A bar A is clamped to the end of the wire, and is not removed during the experiment. The ends of the bar are shaped to fit two slots cut in the walls of a thin tube B, which is of large diameter, and when placed in position the tube is held by the bar coaxial with the wire. Three such tubes are supplied, of different lengths, and all cut from the same stock tube, so that all have the same radius of gyration (p. 182) but different moments of inertia. The

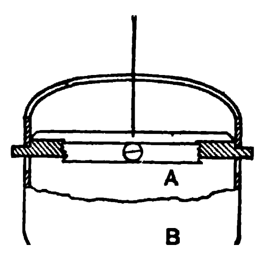



FIG. 254.—Torsional oscillator.

period is found experimentally for each tube in succession, and $\lambda$ is calculated as follows :

Let $M_1$, $M_2$, $M_3$ be the masses of the tubes, found by weighing ; I the moment of inertia of the bar, and $I_1$, $I_2$, $I_3$ the moments of inertia of the cylinders ; R the mean radius of the tubes ; $T_1$, $T_2$, $T_3$ the three periods. $\lambda$ is the same for each experiment since the wire has not been changed. Then

$$T_1^2 = 4\pi^2 (I_1 + I)/\lambda. \quad \text{.........(2)}$$

$$T_2^2 = 4\pi^2 (I_2 + I)/\lambda. \quad \text{.........(3)}$$

$$T_3^2 = 4\pi^2 (I_3 + I)/\lambda. \quad \text{.........(4)}$$

(2) and (3) give

$$T_1^2 - T_2^2 = \frac{4\pi^2}{\lambda}(I_1 - I_2),$$

$$\lambda = 4\pi^2\left(\frac{I_1 - I_2}{T_1^2 - T_2^2}\right) = 4\pi^2\left(\frac{M_1R^2 - M_2R^2}{T_1^2 - T_2^2}\right)$$

$$= 4\pi^2R^2\left(\frac{M_1 - M_2}{T_1^2 - T_2^2}\right). \quad \text{.........(5)}$$

It will be seen that this calculation is easy, since the moment of inertia of the bar does not enter.

Similarly (2) and (4) give

$$\lambda = 4\pi^2R^2\left(\frac{M_1 - M_3}{T_1^2 - T_3^2}\right), \quad \text{.........(6)}$$

and (3) and (4) give

$$\lambda = 4\pi^2R^2\left(\frac{M_2 - M_3}{T_2^2 - T_3^2}\right). \quad \text{.........(7)}$$

The use of three cylinders thus enables corroborative results to be calculated.

EXPT. 38.—**Torsional oscillations.** By use of the apparatus described, find $\lambda$ for each of the wires supplied (copper, brass, steel, etc.). Be careful to record the material of the wire under experiment, as well as the diameter and length of the wire.

**Compound pendulum.**—A **compound pendulum** consists of any heavy rigid body vibrating about a fixed horizontal axis under the action of gravity In the position of static equilibrium, the centre of gravity G (Fig. 255) and the axis C are in the same vertical CY, and the amplitude of vibration of CG should be through a small angle only on each side of this vertical.

The compound pendulum in Fig. 256 oscillates through a small angle $\alpha$ on each side of CY, and at the instant considered makes an angle $\theta$ with CY. Let the mass of the pendulum be M, then the weight $Mg$ acts vertically through the centre of gravity G which is at a distance $\bar{y}$ from C. The couple producing angular acceleration is

$$L = Mg \times GK = Mg\bar{y} \sin\theta.$$

Since $\theta$ is assumed to be small, $\theta$ and $\sin\theta$ will be nearly equal, hence

$$L = Mg\bar{y}\theta. \quad \ldots\ldots(1)$$

Thus the couple is proportional to $\theta$ and therefore the oscillations are simple harmonic, and the period is given by

$$T = 2\pi\sqrt{\frac{I_C}{\lambda}} \text{ (p. 206).} \quad \ldots\ldots(2)$$

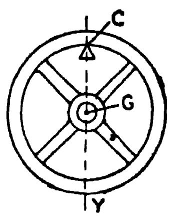



FIG. 255.—Compound pendulum.

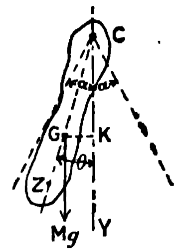



FIG. 256.

The value of $\lambda$ is the couple when the displacement is one radian, and is therefore obtained by putting $\theta = 1$ in (1) above ; thus

$$\lambda = Mg\bar{y}.$$

Also $I_C$ may be written $Mk_C^2$, where $k_C$ is the radius of gyration with respect to the axis C. Substituting these values in (2) we have

$$T = 2\pi\sqrt{\frac{Mk_C^2}{Mg\bar{y}}} = 2\pi\sqrt{\frac{k_C^2}{g\bar{y}}}. \quad \ldots\ldots(3)$$

Comparing this result with the period of a simple pendulum, viz.

$$T = 2\pi\sqrt{\frac{l}{g}}, \text{ (p. 201)}$$

we see that, if a simple pendulum be constructed having a length $l$ given by

$$l = \frac{k_C^2}{\bar{y}}; \quad \ldots\ldots(4)$$

then the simple pendulum will have the same period as the compound pendulum, and may be called the **equivalent simple pendulum.**

EXAMPLE.—A solid disc, 2 feet in diameter, oscillates under gravity about a horizontal axis at the rim and perpendicular to the plane of the disc. Find the length of the equivalent simple pendulum and the frequency.

$$I_C = I_G + MR^2 = 0{\cdot}5MR^2 + MR^2 = 1{\cdot}5MR^2.$$

$$\therefore\ k_C^2 = 1{\cdot}5R^2 = 1{\cdot}5 \text{ ft.}^2$$

$$\bar{y} = R = 1 \text{ foot.}$$

$$l = \frac{k_C^2}{\bar{y}} = 1{\cdot}5 \text{ feet.}$$

$$T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{1\cdot 5}{32\cdot 2}} = 1\cdot 356 \text{ sec.}$$

$$n = \frac{1}{T} = 0\cdot 777 \text{ oscillation per sec.}$$

**Centre of oscillation.**—In Fig. 256 let CG be produced to Z so that CZ equals the length of the equivalent simple pendulum $l$, then Z is called the **centre of oscillation.** This point may be defined by stating that the compound pendulum would oscillate in the same time if its whole mass were concentrated at Z. Denote GZ by $z$, then from (4),

$$k_C^2 = l\bar{y} = (\bar{y} + z)\bar{y}.$$

Also, $$k_C^2 = k_G^2 + \bar{y}^2, \text{ (p. 179)}$$

$$\therefore\ k_G^2 + \bar{y}^2 = (\bar{y} + z)\bar{y},$$

$$\therefore\ k_G^2 = z\bar{y}, \quad\quad(5)$$

and, from above, $$k_C^2 = z\bar{y} + \bar{y}^2. \quad\quad(6)$$

The period given in (3) can therefore be written

$$T = 2\pi\sqrt{\frac{z\bar{y} + \bar{y}^2}{g\bar{y}}} = 2\pi\sqrt{\frac{z + \bar{y}}{g}}. \quad\quad(7)$$

Suppose that the compound pendulum is inverted and suspended at Z (Fig. 257). The period is now

$$T = 2\pi\sqrt{\frac{k_Z^2}{gz}}.$$

Also, $$k_Z^2 = k_G^2 + z^2 = z\bar{y} + z^2 \text{ (from (5))}$$

$$\therefore\ T = 2\pi\sqrt{\frac{z\bar{y} + z^2}{gz}} = 2\pi\sqrt{\frac{\bar{y} + z}{g}}. \quad\quad(8)$$

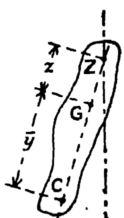



FIG. 257.—Inverted compound pendulum.

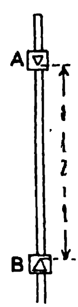



FIG. 258.—Kater pendulum.

Comparison of (7) and (8) shows that the expressions are identical, therefore the periods are the same whether the pendulum is suspended from C, or from the centre of oscillation Z.

The determination of the value of $g$ may be carried out by use of a

simple pendulum (p. 205); there is, however, difficulty in measuring $l$ accurately in a practical simple pendulum. This difficulty disappears in the **Kater compound pendulum**, which consists of a metal bar with two adjustable knife edges A and B (Fig. 258). The distance between the knife edges is adjusted until the frequency is the same whether the pendulum is suspended at A, or inverted and suspended at B. The length $l$ of the equivalent simple pendulum is then the distance between the knife edges, and this can be measured accurately The period is determined by experiment and $g$ is thus calculated from

$$g = \frac{4\pi^2 l}{T^2}.$$

**Experimental determination of moments of inertia.**—The properties of the compound pendulum enable the moment of inertia of a complex body, such as a flywheel, to be obtained easily. Thus, in Fig. 255, a small flywheel is suspended on a knife edge at C parallel to the axis through G about which the flywheel will subsequently rotate. Care must be taken to make the flywheel oscillate in the vertical plane which is perpendicular to the knife edge, and to keep the amplitude small. By observing the time taken to execute 100 complete oscillations, the time T of one oscillation is found and hence the length $l$ of the equivalent simple pendulum is calculated from

$$l = \frac{T^2 g}{4\pi^2}.$$

The distance $\bar{y}$ from C to the centre of gravity G is measured, and the wheel is weighed to determine its mass M. Then from (4),

$$k_C^2 = l\bar{y} = k_G^2 + \bar{y}^2,$$

$$\therefore\ k_G^2 = l\bar{y} - \bar{y}^2 = (l - \bar{y})\bar{y},$$

And,

$$I_G = Mk_G^2 = M(l - \bar{y})\bar{y}$$

$$= M\left(\frac{T^2 g}{4\pi^2} - \bar{y}\right)\bar{y}. \quad \text{......(9)}$$

**Centre of percussion.**—Let a rigid body of mass M be capable of rotation freely about a horizontal axis at C (Fig. 259 (*a*)) and let a horizontal impulsive force P be applied through a point Z. In general, an impulse will be communicated to the axis C. There is, however, one position of Z for which there is no impulse at C; this position is called the **centre of percussion**. Suppose the axis C to be removed, and transfer P from Z to the centre of mass G (Fig. 259 (*b*)). This will necessitate the introduction of a couple having a moment P × GZ (p. 115). Owing to the force P acting through G every particle of the body will have an acceleration $a_1$ towards the left and given by

$$a_1 = \frac{P}{M}, \quad \text{......(10)}$$

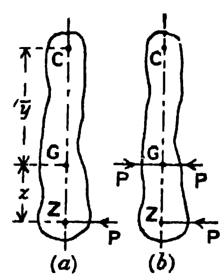



FIG. 259.—Centre of percussion.

and C has this acceleration imparted to it. The couple produces a clockwise angular acceleration $\phi$ given by

$$P \times GZ = I_G\phi = Mk^2{}_G\phi,$$

$$\therefore \phi = \frac{P \times GZ}{Mk_G{}^2}.$$

This angular acceleration gives C a linear acceleration $a_2$ towards the right, and

$$a_2 = \phi \times CG$$

$$= \frac{P \times GZ \times CG}{Mk_G{}^2}. \quad \text{.............................(11)}$$

The accelerations $a_1$ and $a_2$ are of opposite senses, and if they are equal there will be no impulse at C ; hence the condition is, from (10) and (11),

$$\frac{P}{M} = \frac{P \times GZ \times CG}{Mk_G{}^2},$$

or, $$z\bar{y} = k_G{}^2. \quad \text{.........................................(12)}$$

This is identical with the result (5) for the compound pendulum, and it therefore follows that the **centres of oscillation and percussion coincide.**

## Exercises on Chapter XVI

**1.** A body having a mass of 20 pounds revolves in a circular path of 9 inches radius with a velocity of 40 feet per second. Find the centrifugal force.

**2.** A small wheel revolves 24,000 times per minute. There is a body having a mass of 0·05 pound fastened to the wheel at a radius of 4 inches. Find the centrifugal force.

**3.** Assuming that the earth rotates once in 24 hours, and that the equatorial diameter is 8000 miles, find the centrifugal force acting on a person having a mass of 150 pounds when at the equator.

**4.** A cylinder has equal masses of 10 pounds each attached to its ends at radii of 9 inches. The distance between the masses, parallel to the axis of the cylinder, is 12 inches. Looking at the end of the cylinder, both masses appear to be on the same diameter, on opposite sides of the centre. Calculate the rocking couple when the angular velocity is $10\pi$ radians per second.

**5.** A railway coach, mass 20 tons, runs round a curve of 1600 feet radius at a speed of 45 miles per hour. Calculate the centrifugal force. If both rails are on the same level, 5 feet apart centre to centre, and if the centre of mass of the coach is 6 feet above rail level, find the resultant force on each rail.

**6.** An oval track for motor cycles has a minimum radius of 80 yards, and has to be banked to suit a maximum speed of 65 miles per hour. Find the slope of the cross-section at the places where the minimum radii occur.

7. A bicycle and rider together have a mass of 180 pounds. Find the angle which the machine must make with the horizontal in travelling round a curve of 12 feet radius at 8 miles per hour. At this speed, what frictional force must the ground exert on the wheels if no side slip occurs? What is the minimum safe value of the coefficient of friction?

8. A point describes simple harmonic vibrations. If the period is 0·3 second and the amplitude 1 foot, find the maximum velocity and the maximum acceleration.

9. A body having a mass of 4 grams executes simple harmonic vibrations. The force acting on the body when the displacement is 8 cm. is 24 grams weight. Find the period. If the maximum velocity is 500 cm. per second, find the amplitude and the maximum acceleration.

10. A simple pendulum beats quarter-seconds in a place where $g = 32{\cdot}18$ feet per second per second. Find its length. If the pendulum is taken to a place where $g = 32{\cdot}2$ feet per second per second, how many seconds per day would it gain or lose?

11. Two simple harmonic vibrations, A and B, of equal periods and differing in phase by $\pi/2$, are impressed on the same particle. The amplitudes of A and B are 4 and 6 inches respectively. Find the amplitude of the resulting vibration and its phase difference from A.

12. In a conical pendulum find the height in feet of the cone of revolution for velocities of 20, 40, 60, 80, 100, 120 revolutions per minute. Plot a graph showing the relation of the height and the revolutions per minute.

13. The height of a conical pendulum is 8 inches and the arm is 12 inches long. Find the period. If the mass at the end of the arm is 2 pounds, find the pull in the arm. Find the revolutions per minute at which the arm will make 45° with the axis of revolution.

14. Find the change in the height of the cone of revolution of a simple unloaded governor when the speed changes from 60 to 62 revolutions per minute.

15. In a loaded governor the mass at the end of each arm is 2 pounds. The arms are each 8 inches long. The height of the cone of revolution has to be 5 inches at 180 revolutions per minute. Find the load which must be placed on the sleeve.

16. In the governor given in Question 15, the heights between which the governor works are 5·5 and 4·5 inches. Find the maximum and minimum speeds of revolution.

17. A train is travelling round a curve of 500 feet radius at a speed of 30 miles per hour. The distance between centres of rails is 3 ft. 9 in. If the resultant force on the train is to be perpendicular to the line joining the tops of the rails, find how much the outer rail must be raised above the inner.

18. The roadway of a bridge over a canal is in the form of a circular arc of radius 50 ft. What is the greatest velocity (in miles per hour) with which a motor cycle can cross the bridge without leaving the ground at the highest point?

19. A train is travelling in a curve of 240 yards radius. The centre of gravity of the engine is 6 feet above the level of the rails, and the distance between the centre lines of the rails is 5 feet. Find the speed at which the engine would be just unstable, if the rails are both at the same level.
L.U.

**20.** A motor racing track of radius $a$ is banked at an angle $\alpha$; obtain an equation which will give the speed for which the track is designed. Show that if the speed of a car is one-half this speed there will be a total transverse frictional force of $\frac{3}{4}W \sin \alpha$ between the car and the ground, W being the weight of the car. L.U.

**21.** The period of a simple harmonic motion is $2\pi/p$, and its amplitude is A. Prove that the displacement can be expressed in the form

$$A \cos (pt - \alpha),$$

and find the velocity.

The distance between the extreme limits of the oscillation is 6 inches, and the number of complete oscillations per minute is 100. Calculate the velocity of the point when it is 2 inches from the centre; find also the interval of time from the centre to that point.

**22.** A particle is performing a simple harmonic motion of period T about a centre O, and it passes through a point P with velocity $v$ in the direction OP; prove that the time which elapses before its return to P is $(T/\pi) \tan^{-1}(vT/2\pi OP)$. L.U.

**23.** A particle moves with simple harmonic motion; show that its time of complete oscillation is independent of the amplitude of its motion. The amplitude of the motion is 5 feet and the complete time of oscillation is 4 secs.; find the time occupied by the particle in passing between points which are distant 4 feet and 2 feet from the centre of force and are on the same side of it.

**24.** A weight of 5 lb. is tied at the end of an elastic string, whose other end is fixed, and is in equilibrium when the string is of length 14 inches, its unstretched length being 12 inches. The weight is pulled gently down, through another inch, and then let go; find the time of the resulting oscillation. L.U.

**25.** Show that the vertical distance of the bob of a conical pendulum beneath the fixed end of the string depends only upon the number of revolutions of the pendulum per sec. If the mass of the bob is 4 pounds, and the length of the string is 2 ft., find the maximum number of revolutions per second of the pendulum when the greatest tension that can with safety be allowed in the string is 40 lb. weight. L.U.

**26.** Prove that the restoring force acting on a simple pendulum is proportional to the angle through which it is displaced from the equilibrium position, provided this angle be small.

Describe also a method of verifying the above result by experiment.

**27.** A simple pendulum, 10 feet long, swings to and fro through a distance 2 inches. Find its velocity at its lowest point, its acceleration at its highest point, and the time of an oscillation, calculating each result numerically in foot and second units.

**28.** Investigate the time of revolution of a conical pendulum.

A ball, of mass one pound, describes a horizontal circle attached to two cords, the other ends of which are fixed to two points in the same vertical line. The cords are each of length 3 feet, and are at right angles to one another. If the ball makes 100 revolutions a minute, compute the tension of each cord in pounds weight.

**29.** Two equal light rods, AB and BC, are freely jointed to a particle of mass $m$ at B; the end A of the rod AB is pivoted to a fixed point A, and the end C of BC is freely jointed to a smooth ring of mass $m$, which can slide on a smooth vertical rod AC. Show that, when C is below A and the

mass at B is describing a horizontal circle with uniform angular velocity $\omega$, $\cos \alpha = 3g/l\omega^2$, where $\alpha$ is the inclination of the rods to the vertical and $l$ is the length of either rod. L.U.

**30.** Show that a body moving with uniform velocity $v$ in a circle of radius $r$ has acceleration equal to $v^2/r$ directed towards the centre. Hence explain why a man riding a bicycle on a curved path has always to bend his body inwards towards the centre of the path.

**31.** A solid cylinder is 2 inches in diameter and has a mass of 3·64 pounds; it is clamped coaxially to the lower end of a wire; the upper end is firmly attached to a fixed support. If the cylinder executes torsional oscillations having a period of 0·82 second, find the couple required to twist the wire through one radian.

**32.** A brass bar is 10 cm. long, 2 cm. wide and 0·5 cm. thick; the density is 8·6 grams per c.c. The lower end of a wire is attached firmly to the centre of one of the 10 cm. × 2 cm. faces and the bar executes torsional oscillations. A separate experiment showed that a couple of 8·2 gram-wt.-cm. was required to twist the wire through $\frac{1}{2}\pi$ radians. Find the period.

**33.** In an experiment on torsional oscillations, a copper wire 0·047 inch in diameter and 73·5 cm. long was used, the apparatus being that shown in Fig. 254. The cylinders had a mean diameter of 11·27 cm. and the masses were 471, 237 and 120 grams respectively. The corresponding periods were found to be 2·4, 1·77 and 1·35 seconds. Calculate the three values of the couple required to twist the wire through one radian, and state the mean value.

**34.** A light rod has a mass $m_1$ attached to one end and another mass $m_2$ attached to the other end. If the arrangement is thrown whirling into the air prove that, apart from the motion of translation, the masses whirl about the centre of mass of the system.

**35.** A mass $m$ is attached to one end of a cord of length $l$, and the other end is attached to a fixed peg. If the mass whirls in a vertical circle of which the peg is the centre with an angular velocity $\omega$ prove that (a) the least value of $\omega$ which will keep the cord taut is $\sqrt{g/l}$; (b) the maximum pull in the cord is $m(g + \omega^2 l)$.

**36.** Describe an accurate pendulum method of measuring $g$, the acceleration due to gravity. What is the general nature of the variation of $g$ with latitude?

If the earth's rotation ceased, what would be the change of $g$ at the equator? (Diameter of the earth, 8000 miles.) C.W.B., H.C.

**37.** Show that when a particle is moving uniformly round a circle of radius $r$ with velocity $v$, the particle has an acceleration of $v^2/r$ directed towards the centre of the circle.

An indiarubber band of mass 4 gm., when stretched round a horizontal pulley of 10 cm. radius, has a tension of 3000 dynes. Find the number of turns per second made by the pulley when the band first ceases to press against it. C.W.B., H.C.

**38.** If a body X moves to and fro in a straight line between two fixed points A and B in such a way that its kinetic energy is always proportional to the product AX . XB, show that it executes simple harmonic motion.

Calculate the maximum speed of the prong of a tuning fork of frequency 500 when vibrating with an amplitude of 1 mm. What fraction of its energy of vibration has it lost when the amplitude has fallen to $\frac{1}{2}$ mm.? C.W.B., H.C.

**39.** A point is moving in a straight line with simple harmonic motion. Its velocity has the values 5 feet per second and 4 feet per second when its distances from the mean position are 2 feet and 3 feet respectively. Find the length of its path and the period of its motion, taking $\pi = 3\cdot1416$. Determine what fraction of the period is occupied in passing between the two points if they are on opposite sides of the mean position.
J.M.B., H.S.C.

**40.** A light arm CB, of length $a$, is freely pivoted at its end C which is fixed, and carries at B a mass $m$; the arm is maintained in a horizontal position by a string attached to B and to a point A fixed vertically above C at a distance $b$ from it. Find the magnitude and direction of the stress in CB when CB is revolving about the vertical at the uniform rate of $n$ revolutions per second. J.M.B., H.S.C.

**41.** A circular motor track has a radius of 800 feet, and is banked at an angle of 30°. Find the speed at which a car must go round the track so that there shall be no tendency to side-slip, and find the pressure of the car on the track at this speed if the mass of the car is 15 cwt. J.M.B., H.S.C.

**42.** A particle describes a circle of radius $a$ in a vertical plane, moving round at the end of a taut string fastened at the centre. If the velocity of the particle as it swings past the lowest point of the circle is $v_0$, prove that the tension in the string when it is inclined at an angle $\theta$ to the radius to the lowest point is $m(3g\cos\theta - 2g + v_0^2/a)$. If $5ga > v_0^2 > 2ga$, prove that the string becomes slack before the particle reaches its highest point. What happens (i) when $v_0^2 > 5ga$, and (ii) when $v_0^2 < 2ga$? J.M.B., H.S.C.

**43.** A compound pendulum consists of a thin uniform bar 100 cm. in length. Find the length of the equivalent simple pendulum when the axis of suspension is (*a*) at one end of the bar; (*b*) at 1 cm. from the centre of the bar. Compare the periods of these two cases.

**44.** A hoop has a diameter of 3 feet and is made of thin wire. The hoop vibrates under the action of gravity about an axis passing through the wire and perpendicular to the plane of the hoop. Find the period.

**45.** A pendulum is constructed of a brass bar 110 cm. long × 2·5 cm. × 0·5 cm. and a brass rectangular block 12 cm. × 12 cm. × 4 cm. through which the bar passes symmetrically (Fig. 260). The block is fixed to the bar with its centre 9 cm. from one end, and the axis of suspension C is 1 cm. from the other end. Find the length of the equivalent simple pendulum and the period. Density of brass $= 8\cdot6$ grams per c.c.; $g = 981$ cm. per sec.$^2$.

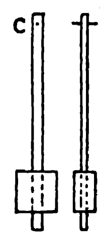


FIG. 260.

**46.** A small flywheel, mass 18 pounds, is mounted as shown in Fig. 255. The distance CG is 7 inches and the length of the equivalent simple pendulum is found to be 12·5 inches. Find the moment of inertia of the wheel about an axis through G and perpendicular to the plane of the wheel in pound and foot units.

**47.** Take a cricket bat, or a golf club (driver), and discuss the design of the appliance as influenced by the properties of the centre of percussion.

**48.** If you were provided with a large circular metal sheet of uniform thickness and with arrangements for suspending it so that it could oscillate freely, under the influence of gravity, in its own plane about various horizontal axes perpendicular to that plane, how would you expect the time

of oscillation to vary with the distance of the axis from the centre of the circular sheet? Draw a rough graph of the relation and offer some theoretical explanation. J.M.B., H.S.C.

**49.** A helical spring is hung vertically from its upper end and a mass $m$ is attached to the lower end. The unstretched length of the spring is $a$ and the addition of $m$ causes the length to become $b$. The mass is pulled downwards slightly and then released. Neglect the mass of the spring and show that the mass $m$ will execute simple harmonic vibrations having a periodicity the same as that of a simple pendulum whose length is $(b-a)$.

**50.** A body can rotate about a fixed axis which does not pass through the centre of mass. Describe an experimental method of determining the distance from the fixed axis to the centre of percussion. State what observations must be made and how the result is calculated.

**51.** A thin uniform rod of length $l$ hangs freely from an axis at one end and perpendicular to the rod. Show that the centre of percussion is at a distance $\frac{2}{3}l$ from the axis.

**52.** A uniform disc, radius $r$, can rotate about an axis at its rim and perpendicular to the plane of the disc. Show that the centre of percussion is at a distance $1{\cdot}5r$ from the axis.

**53.** A thin uniform rod, mass $m$, length $l$, is attached at one end to the surface of a solid ball so that the axis of the rod passes through the centre of the ball. The ball has a mass $\mathsf{M}$ and radius $\mathsf{R}$. The arrangement is suspended freely from an axis at the other end of the rod. Show that the distance from the axis to the centre of percussion is

$$[\tfrac{1}{3}ml^2 + \mathsf{M}\{1{\cdot}4\mathsf{R}^2 + l(l+2\mathsf{R})\}]/\{\tfrac{1}{2}ml + \mathsf{M}(l+\mathsf{R})\}.$$

Find the distance if $\mathsf{M} = 1000$ grams, $m = 100$ grams, $l = 50$ cm., $\mathsf{R} = 10$ cm.

**54.** A body is secured to an axle which rotates in fixed bearings. Explain with references to sketches, the conditions which must be fulfilled if the bearings experience no disturbance due to centrifugal action.

# CHAPTER XVII

## IMPACT

**Direct impact.**—**Direct impact** occurs when two bodies are both travelling in the straight line joining their centres of mass before collision, or when a moving body impinges normally on a fixed surface. It is not possible to state the precise magnitude of the stress between two bodies, A and B, at any instant during impact, but we may say that whatever action A exerts on B, at the same instant B exerts an equal opposite reaction on A. Also these actions are maintained during the same interval of time. Thus a diagram, showing the relation of the action F which A exerts on B at any instant $t$ seconds after the commencement of impact, would be similar and equal to a diagram showing the reactions which B exerts on A. The area of such a diagram represents the change in momentum of the body (p. 64); hence, since the areas are equal, we may say that the momentum acquired by one body is equal and opposite to that lost by the other body during the impact. It follows from this that **the total momentum before impact must be equal to the total momentum after impact is completed.**

**Inelastic and elastic bodies.**—The motion of the bodies after collision depends greatly on the degree of elasticity possessed by them. A body having no elasticity makes no effort whatever to recover its original form and dimensions. For example, deformation of a plastic substance like putty, which is practically inelastic, progresses so long as a force is exerted on it, and the putty retains the shape it possessed at the instant of the removal of the force. When two such bodies collide with direct impact, force between them ceases at the instant when their centres of mass cease to approach each other. Hence there is no tendency for the bodies to separate, and they continue to move as one body. In other words, the relative velocity after collision is zero.

In the case of elastic, or partially elastic bodies, the force does not cease at the instant of closest approach of the centres of mass. The effort which the bodies make to recover their original dimensions causes the action and reaction to continue, with the result that there is a second period during the impact, in which the centres of mass are receding from each other. Finally, the efforts to recover the original dimensions cease, and at this instant the bodies separate, and continue to move separately. Experiment shows that, roughly, the relative velocity after collision bears a definite ratio to the relative velocity

before collision, and is of opposite sense. The value of this ratio differs for different materials; it is called the **coefficient of restitution.**

In direct impact (Fig. 261), let

$u_1$ = the velocity of the body A before impact.
$u_2$ = the velocity of the body B before impact.
$v_1$ = the velocity of the body A after impact.
$v_2$ = the velocity of the body B after impact.
$e$ = the coefficient of restitution.

Then, Relative velocity of approach $= u_1 - u_2$,
Relative velocity of separation $= v_2 - v_1$,

and,
$$e = \frac{v_2 - v_1}{u_1 - u_2}.$$

The coefficient of restitution has values about 0·95 for glass and about 0·2 for lead. Modern experiments indicate that the value of $e$ may differ considerably for different parts of the surface of the same body. It is also well known that, if two metal bodies impinge twice, so that the same parts of their surfaces come into contact on both occasions, the hardness of the surfaces has been so altered during the first impact that a different value of the coefficient of restitution is apparent during the second impact.

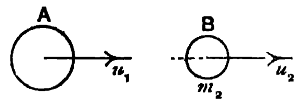



FIG. 261.—Direct impact.

**Direct impact of inelastic bodies.**—In Fig. 262 ($a$) two inelastic bodies of masses $m_1$ and $m_2$, and velocities $u_1$ and $u_2$, are about to experience direct impact. $u_1$ is greater than $u_2$. We have (p. 218)

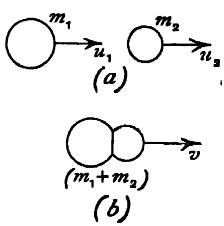



FIG. 262. — Direct impact of inelastic bodies.

Total momentum before impact

= total momentum after impact;

$$\therefore\ m_1u_1 + m_2u_2 = (m_1 + m_2)v,$$

where $v$ is the common velocity after impact (Fig. 262 ($b$));

$$\therefore\ v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2}. \quad \text{..............(1)}$$

If $u_2$ is of the sense opposite to that of $u_1$, then call $u_2$ negative; hence

$$v = \frac{m_1u_1 - m_2u_2}{m_1 + m_2}. \quad \text{..............................(2)}$$

In general,
$$v = \frac{m_1u_1 \pm m_2u_2}{m_1 + m_2}. \quad \text{..............................(3)}$$

$v$ will have the same or the opposite sense to $u_1$, according as the result in (3) is positive or negative.

Since work has been done in deforming the bodies, and there has been no recovery, it follows that energy has been wasted during the collision. The energy wasted may be calculated as follows:

$$\text{Before impact, the total kinetic energy} = \frac{m_1u_1^2}{2} + \frac{m_2u_2^2}{2}. \quad \ldots\ldots(4)$$

$$\text{After impact, the total kinetic energy} = \frac{(m_1 + m_2)v^2}{2}. \quad \ldots\ldots\ldots(5)$$

Hence, $$\text{Energy wasted} = \left(\frac{m_1u_1^2}{2} + \frac{m_2u_2^2}{2}\right) - \frac{(m_1 + m_2)v^2}{2}.$$

Inserting the value of $v$ from (3), we have

$$\text{Energy wasted} = \left(\frac{m_1u_1^2 + m_2u_2^2}{2}\right) - \left(\frac{m_1 + m_2}{2}\right)\left(\frac{m_1u_1 \pm m_2u_2}{m_1 + m_2}\right)^2$$
$$= \frac{m_1u_1^2 + m_2u_2^2}{2} - \frac{(m_1u_1 \pm m_2u_2)^2}{2(m_1 + m_2)}.$$

By squaring and reducing to the simplest form, we obtain

$$\text{Energy wasted} = \frac{m_1m_2}{2(m_1 + m_2)}(u_1 \mp u_2)^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Now $(u_1 - u_2)$ is the relative velocity of approach before impact if both bodies are moving in the same sense, and $(u_1 + u_2)$ is the relative velocity if the senses of the velocities are opposite. Hence **the wasted energy is proportional to the square of the relative velocity of approach.**

**Direct impact of bodies having perfect elasticity.**—Perfect elasticity implies not only perfect recovery of shape and original dimensions, but also perfect restitution of the energy expended during the deformation period. Hence no energy is wasted in the impact of perfectly elastic bodies.

To avoid complications, let the bodies be smooth spheres and let the impact be direct. Let $u_1$ and $u_2$ be the velocities before collision, and let $v_1$ and $v_2$ be the velocities after collision (Fig. 263). As before, we have

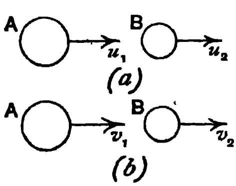



FIG. 263.—Direct impact of elastic bodies.

Total momentum before collision
= total momentum after collision;

$$\therefore\ m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2. \quad \ldots\ldots(7)$$

Also, Total energy before collision
= total energy after collision;

$$\therefore\ \frac{m_1u_1^2}{2} + \frac{m_2u_2^2}{2} = \frac{m_1v_1^2}{2} + \frac{m_2v_2^2}{2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

From (8), $$m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2),$$
$$m_1(u_1 - v_1)(u_1 + v_1) = m_2(v_2 - u_2)(v_2 + u_2). \quad \text{...(8')}$$
From (7), $$m_1(u_1 - v_1) = m_2(v_2 - u_2).$$
Hence, from (8'), $$u_1 + v_1 = v_2 + u_2;$$
$$\therefore\ u_1 - u_2 = v_2 - v_1. \quad \text{...(9)}$$

This result indicates that the relative velocity of approach is in this case equal to the relative velocity of separation; in other words, the coefficient of restitution for perfectly elastic bodies is unity.

In using these and the following equations and in interpreting the results, velocities having the same sense as $u_1$ should be denoted positive, and those of opposite sense, negative; negative results indicate velocities having senses opposite to $u_1$.

Supposing the masses to be equal, then, from (7):
$$u_1 + u_2 = v_1 + v_2.$$
And from (9): $$u_1 - u_2 = v_2 - v_1;$$
$$\therefore\ 2u_1 = 2v_2;$$
$$\therefore\ u_1 = v_2,$$
and $$u_2 = v_1.$$

It therefore follows that in the direct collision of perfectly elastic spheres having equal masses, the spheres interchange velocities during impact.

**Direct impact of imperfectly elastic spheres.**—Reference is made again to Fig. 263. As before, we have:

Total momentum before impact = total momentum after impact.
$$\therefore\ m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2. \quad \text{...(10)}$$
Also, $$e = \frac{v_2 - v_1}{u_1 - u_2} \text{ (p. 219)};$$
$$\therefore\ eu_1 - eu_2 = v_2 - v_1. \quad \text{...(10')}$$
Multiplying this by $m_2$:
$$em_2u_1 - em_2u_2 = m_2v_2 - m_2v_1. \quad \text{...(11)}$$
From (10) and (11),
$$(m_1 - em_2)u_1 + (1 + e)m_2u_2 = (m_1 + m_2)v_1;$$
$$\therefore\ v_1 = \frac{(m_1 - em_2)u_1 + (1 + e)m_2u_2}{m_1 + m_2}. \quad \text{...(12)}$$
Multiplying (10') by $m_1$ gives
$$em_1u_1 - em_1u_2 = m_1v_2 - m_1v_1. \quad \text{...(13)}$$
From (10) and (13),
$$(1 + e)m_1u_1 + (m_2 - em_1)u_2 = (m_1 + m_2)v_2;$$
$$\therefore\ v_2 = \frac{(m_2 - em_1)u_2 + (1 + e)m_1u_1}{m_1 + m_2}. \quad \text{...(14)}$$

**Impact of a smooth sphere on a smooth fixed plane.**—It is not possible to realise a plane absolutely fixed in space; what is meant by a fixed plane is one fixed to the earth. The mass of the body against which an elastic sphere collides is then very large as compared with that of the sphere, and its velocity after impact may be taken as equal to its velocity before impact. Direct impact occurs when the line of motion of the sphere is normal to the fixed plane.

In **direct impact**, if the sphere and fixed plane are either or both inelastic, then the sphere will not rebound. If both sphere and plane are perfectly elastic, then the sphere rebounds with a velocity equal and opposite to that which it possessed before impact. If they are imperfectly elastic, and if the velocities of approach and separation are $u$ and $v$ respectively, then

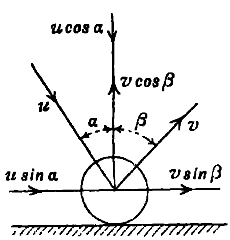


FIG. 264. — Oblique impact of a sphere on a plane.

$$v = eu. \quad \text{.......................(1)}$$

The impact is **oblique** if the line of motion of the sphere prior to impact is inclined to the normal to the plane (Fig. 264). Let $\alpha$ be this angle, and let the sphere leave the plane in a line inclined at $\beta$ to the normal. Let $u$ and $v$ be the initial and final velocities. Resolve these velocities parallel to and perpendicular to the plane. Since both sphere and plane are regarded as being smooth there can be no force parallel to the plane during impact. Hence there can be no change in the component velocity parallel to the plane. Therefore

$$u \sin \alpha = v \sin \beta. \quad \text{.................................(2)}$$

If the coefficient of restitution is $e$, then, from (1):

$$eu \cos \alpha = v \cos \beta. \quad \text{.................................(3)}$$

From (2) and (3):

$$u^2 \sin^2 \alpha + e^2 u^2 \cos^2 \alpha = v^2 (\sin^2 \beta + \cos^2 \beta) = v^2;$$

$$\therefore \; v = u\sqrt{\sin^2 \alpha + e^2 \cos^2 \alpha}. \quad \text{..................(4)}$$

Also, from (2) and (3):

$$\tan \beta = \frac{\tan \alpha}{e}. \quad \text{..................................(5)}$$

If both sphere and plane are perfectly elastic, then $e = 1$, and equations (4) and (5) become

$$v = u, \quad \text{.......................................(6)}$$

$$\tan \beta = \tan \alpha. \quad \text{..................................(7)}$$

Hence in this case the sphere leaves the plane with its initial velocity unaltered in magnitude, and the angles which the initial and final directions of motion make with the normal are equal.

If the sphere be perfectly inelastic, then the whole of the normal component $u \cos \alpha$ disappears, and the sphere will finally slide along the plane with a velocity $u \sin \alpha$.

When a jet of water impinges on a fixed plate (Fig. 265), the impact practically follows the laws of inelastic bodies. The jet spreads out during impact, and the water then slides along the plate.

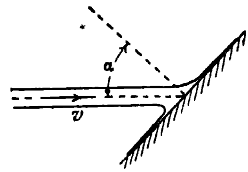



FIG. 265.—Impact of a jet of water.

Let $v$ = the velocity of the jet.

$\alpha$ = the angle between the jet and the normal.

$m$ = the mass of water reaching the plate per second.

Normal component of the velocity $= v \cos \alpha$.

This disappears during impact, hence:

Force acting on the plate = change in momentum per second

$= mv \cos \alpha$.

If A = the cross sectional area of the jet,

$d$ = the density of water,

then, $m = v\mathrm{A}d$;

$\therefore$ Force acting on the plate $= \mathrm{A}dv^2 \cos \alpha$.

**Conservation of momentum.**—This principle asserts that **the total momentum of any system of bodies which act and react on each other remains constant.**

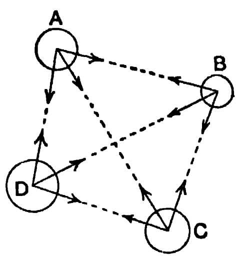



FIG. 266.—Principle of the conservation of momentum.

The truth of the principle will be evident when we consider the equality of the action which one body A exerts on another body B and the reaction which B exerts on A (Fig. 266). These actions continue during the same interval of time; hence whatever momentum B is losing, A is gaining an equal momentum. Hence the total momentum along AB remains constant. Similarly, the total momentum along each of the lines BC, CD, DA, AC and BD remains constant. Therefore the total momentum of the system remains constant. The forces may be caused by gravitational attraction, magnetism, or impact; their nature is immaterial; the important points are their equality, their opposing character and the equality of the times during which they act.

EXPT. 39.—**Coefficient of restitution.** Arrange a tall retort stand A (Fig. 267) with two rings B and C which may be clamped at different heights. D is a massive block of cast-iron or steel. A small steel ball, $\frac{1}{4}$ inch to $\frac{1}{2}$ inch in diameter (these can be obtained from any cycle dealer), is dropped from the level of B and rebounds from D. The ring C is adjusted until it is found that the ball reaches its level in the first rebound. Measure $h_1$ and $h_2$.

Then, keeping the ring C in its initial position, B is shifted to a position below C, and the ball is dropped from the level of C. B is adjusted until it is found that the ball rebounds to its level. In this way are found the heights $h_1$, $h_2$, $h_3$, etc., of successive rebounds.

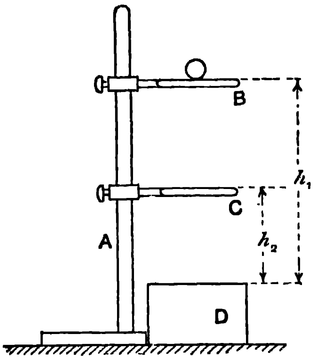



FIG. 267.—Apparatus for determining the coefficient of restitution.

In the first drop,

the velocity of approach $= u_1 = \sqrt{2gh_1}$,

and the velocity of separation $= v_1 = \sqrt{2gh_2}$.

In the second drop,

the velocity of approach $= u_2 = \sqrt{2gh_2}$,

and the velocity of separation $= v_2 = \sqrt{2gh_3}$.

The velocities for the succeeding drops may be calculated in the same way. Now

$$e = \frac{\text{velocity of separation}}{\text{velocity of approach}};$$

$$\therefore \; e_1 = \frac{v_1}{u_1} = \frac{\sqrt{2gh_2}}{\sqrt{2gh_1}} = \sqrt{\frac{h_2}{h_1}},$$

also, $$e_2 = \frac{v_2}{u_2} = \frac{\sqrt{2gh_3}}{\sqrt{2gh_2}} = \sqrt{\frac{h_3}{h_2}}.$$

Similarly, $$e_3 = \sqrt{\frac{h_4}{h_3}}.$$

Work out these values of $e$ from the experimental values of $h_1$, $h_2$, etc. Are they in fair agreement? What is the mean value of $e$? What is the maximum error in the value of $e$ stated as a percentage on the mean value?

EXPT. 40.—**Ballistic pendulum.** In the Hicks's form of this apparatus (Fig. 268) two platforms, A and B, are each suspended from supports by

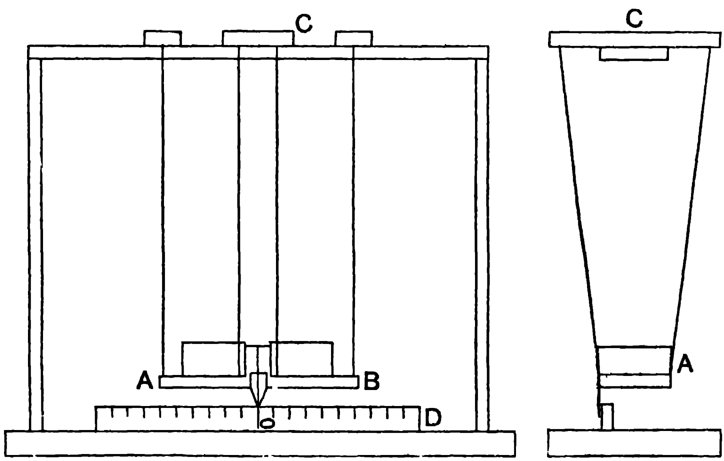



FIG. 268.—Hicks's ballistic pendulum.

means of four threads. As seen in the front elevation, the threads appear vertical; in the end elevation the threads spread as they approach the upper support. The platforms just touch when hanging freely, and in this position the pointer which each carries is at zero on the scale D. The platforms are of equal mass and can be loaded by placing weights on them. There is a locking contrivance, by means of which the platforms become locked together automatically after impact, and thus move as one body. The suspending threads are about 3 feet in length.

Referring to Fig. 269, in which the bob of a pendulum has been displaced a distance BD from the vertical, and has been raised a height CD from the position of static equilibrium, we have for the velocity $v$ at the instant the bob passes through C when swinging freely

$$v = \sqrt{2g \,.\, \mathrm{CD}}.$$

Now, $\mathrm{CD} \times 2\mathrm{AC} = \mathrm{BD}^2$ nearly;

$$\therefore \quad v = \sqrt{2g \frac{\mathrm{BD}^2}{2\mathrm{AC}}} = \text{a constant} \times \mathrm{BD}.$$

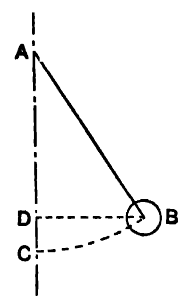


FIG. 269.

Hence the maximum velocity is very nearly proportional to the horizontal displacement. In the Hicks's pendulum we may therefore assume that the maximum velocity of either platform is proportional to the distance through which it has been displaced, as shown by the scale D (Fig. 268).

Place equal masses on the platforms; displace each platform to the same extent and let go. It will be found that the platforms immediately after impact are at rest. This follows from the fact that the momenta immediately before impact were equal and opposite, and hence the total momentum was zero.

Now load A until the total mass is, say, 1 pound, and load B until its total mass is, say, 2 pounds. Displace B through 2 inches, and displace A through 4 inches; again let go and observe what happens at the moment of impact. If the platforms remain at rest, the momenta before impact were equal and opposite. Repeat the experiment, varying the masses of A and B and also the displacements.

## Exercises on Chapter XVII

1. Two inelastic bodies, A and B, moving in the same straight line come into collision. The mass of A is 4 pounds and its velocity is 10 feet per second; the mass of B is 10 pounds and its velocity is 6 feet per second. Find the common velocity after collision. How much energy has been wasted?

2. Answer Question 1, supposing the velocity of B to be $-6$ feet per second.

3. Direct impact occurs between two spheres A and B. The masses are 4 and 3 kilograms respectively, and the velocities are 12 and 8 metres per second respectively. The coefficient of restitution is 0·7. Find the velocities after impact. Find also the energy wasted.

4. Answer Question 3, supposing the velocity of B to be $-8$ metres per second.

5. In Questions 3 and 4, what would be the velocities of A and B after collision, supposing both bodies had been perfectly elastic?

6. A sphere A, having a mass of 10 pounds, experiences direct impact with another sphere B, mass 16 pounds, velocity 12 feet per second. The coefficient of restitution is 0·5. Find the initial velocity of A if it remains at rest after impact; find also the velocity of B after impact.

7. A small steel ball is dropped vertically on to a horizontal fixed steel plane from a height of 9 feet. If the coefficient of restitution is 0·8, find the heights of the first, second, third and fourth rebounds. If the mass of the ball is 0·1 pound, how much energy is wasted during the first three impacts?

8. In Question 7 the ball is dropped vertically from the same height, and the fixed plane is at an angle of 30° to the horizontal. Find the velocity with which the ball leaves the plane. Assume both ball and plane to be smooth.

9. A jet of water having a sectional area of 0·5 square inch and a velocity of 40 feet per second, impinges on a fixed flat plate. Find the force acting on the plate when the jet makes angles of 0, 30, 45, 60 and 90 degrees with it. Plot a graph showing the relation of forces and angles.

10. If a gun of mass M fires horizontally a shot of mass $m$, find the ratio of energy of the recoil of the gun to the energy of the shot.

If a ½-ton gun discharges a 50-pound shot with a velocity of 1000 ft. per sec., find the uniform resistance necessary to stop the recoil of the gun in 6 inches. L.U.

11. State Newton's law of impact, and show how it can be experimentally verified. A smooth sphere of small radius moving on a horizontal table strikes an equal sphere lying at rest on the table at a distance $d$ from a vertical cushion, the impact being along the line of centres and normal to the cushion. Show that if $e$ be the coefficient of restitution between the spheres and between a sphere and the cushion, the next impact between the spheres will take place at a distance $\frac{e^2}{1+e^2} \cdot 2d$ from the cushion.

L.U.

12. What do you understand by Conservation of Momentum? Describe an experimental method of illustrating the conservation of momentum at the impact of two bodies.

13. Define 'impulse' and energy, and give their dimensions in terms of the fundamental units of length, time and mass.

A box of sand, used as a ballistic pendulum, is suspended by four parallel ropes, and a shot is fired into its centre. In one experiment the weight of the box was 1000 lb., the weight of the shot was 10 lb., the length of the ropes was 6 feet, and the displacement of the centre of mass of the box and shot was 4½ feet. What was the velocity of the shot before hitting the box? L.U.

14. A uniform chain, 10 feet long and having a mass of 4 pounds, hangs vertically from an upper support, and its lower end touches the scale-pan of a balance. The upper end is released, and the chain falls into the scale-pan. Find the force acting on the pan at the instant when the last link reaches the pan. Find the energy wasted.

**15.** Explain what is meant by the conservation of momentum and the conservation of energy.

A rifle bullet is fired horizontally into a massive block of wood forming the bob of a pendulum. Describe the measurements which you would make in order to find the velocity of the bullet, and explain how the result is deduced from them. C.W.B., H.C.

**16.** Define momentum and kinetic energy. A mass of 10 pounds forming the bob of a simple pendulum of length 13 ft. is drawn aside to a distance of 5 ft. from the vertical line through the point of suspension and then released. Find the momentum and the kinetic energy of this mass when it reaches its lowest point.

If at this point it strikes and adheres to a mass of 15 pounds at rest which forms the bob of another pendulum, find the velocity with which the two move away together. C.W.B., H.C.

**17.** A particle, mass $m_1$, velocity $v_1$, and another particle, mass $m_2$, velocity $v_2$, move in the same straight line. Show that the total momentum is the same as that of a particle of mass $(m_1+m_2)$ situated at and moving with the centre of mass of the two particles.

Show also that the total kinetic energy of the two particles may be written $\frac{1}{2}(m_1+m_2)v^2+\frac{1}{2}(m_1r_1^2+m_2r_2^2)$ where $v$ is the velocity of the centre of mass and $r_1$ and $r_2$ are the velocities of the particles relative to the centre of mass.

**18.** Using the symbols employed on p. 221 for the direct impact of imperfectly elastic spheres, show that

$$(u_1-v_1)/(u_1-u_2)=m_2(1+e)/(m_1+m_2).$$

**19.** Two spheres of masses $m$, $m^1$ moving with relative velocity V collide directly; prove that the loss of kinetic energy is

$$\frac{mm^1}{2(m+m^1)}(1-e^2)V^2,$$

where $e$ is the coefficient of restitution.

Two spheres of equal mass moving in the same straight line with velocities $u$, $u^1$ collide and rebound, the coefficient of restitution being $\frac{1}{2}$. Prove that exactly half the energy is lost in collision if $(1-\sqrt{2})u=(1+\sqrt{2})u^1$.

(*N.B.*—It is convenient to employ the result obtained in Exercise 18.) C.W.B., H.C.

**20.** State the laws which determine the change of motion of two particles produced by their direct collision.

Two spheres of masses 6 oz. and 8 oz. are moving towards each other in the same line with velocities 10 ft./sec. and 6 ft./sec. respectively and they collide directly. Find their velocities after impact if the coefficient of restitution is 0·75, and determine the loss of energy resulting from the impact. J.M.B., H.S.C.

# CHAPTER XVIII

## HYDROSTATICS

**Definition of a fluid.**—Substances in the **fluid state** are incapable of offering permanent resistance to any forces, however small, tending to change their shape. Fluids are either **liquid** or **gaseous**. Gases possess the property of indefinite expansion and liquids do not. Liquids in a partially filled vessel show a distinct surface not coinciding with any of the walls of the vessel; if this surface is in contact with the air, as would be the case in an open vessel, it is called the **free surface.**

Comparatively small compressive forces cause appreciable alteration in the volume of a gas; liquids show very little change in volume, even when the compressive forces are very great. It may be assumed for our present purposes that liquids are incompressible. This assumption, together with the neglect of changes in volume due to changes in temperature, is equivalent to taking the density of any given liquid to be constant.

The property which differentiates a liquid from a solid is the ability of the former to flow. Some liquids, such as treacle and pitch, flow with difficulty, and are said to be **viscous**; the property is called **viscosity. Mobile** liquids, such as alcohol and ether, flow easily. No fluid is perfectly mobile.

That branch of the subject which treats of fluids at rest is called **hydrostatics.** In **hydrokinetics,** the laws of fluids in motion are discussed. **Pneumatics** deals with the pressure and flow of gases. **Hydraulics** is the branch of engineering which treats of the practical applications of the laws of the pressure and flow of liquids, especially of water.

**Normal stress only can be present in a fluid at rest.**—Change of shape of a body occurs always as a consequence of the application of shearing stresses (p. 128). Hence, if there be shearing stresses present in a fluid, the fluid must be in the act of changing shape, and must therefore be in motion. Therefore **there can be none but normal stresses acting on the boundary surfaces and on any section of a fluid at rest.** Since friction is always evidenced as a force acting tangentially to the sliding surfaces, it follows that there can be no friction in a fluid at rest.

The term **pressure** is given to the normal stress which a fluid applies to any surface with which it is in contact. Pressure is stated in units

of force per unit of area. The dimensions are therefore the same as those of stress, viz.

$$\frac{ml}{t^2} \times \frac{1}{l^2} = \frac{m}{lt^2}, \quad \text{or} \quad ml^{-1}t^{-2}.$$

In general, the pressure of a fluid varies from place to place. The **pressure at a given point** may be defined as follows: Take a small area $a$ embracing the point, and let P be the resultant force which the fluid exerts on $a$. The average value of the pressure on $a$ is P/$a$. The actual value of the pressure at any part on the small area differs from the average value to a small extent only, and the difference will become smaller if $a$ be diminished. Of course, P will then become smaller also. If $a$ be diminished indefinitely, thus approximating to a point, the value of P/$a$ gives the pressure at this point.

Pressures may be stated in dynes, or in grams weight, per square centimetre; for practical purposes the most convenient metric unit is the kilogram weight per square centimetre. In the British system we may use poundals per square foot, or, for practical purposes, pounds weight per square foot, or per square inch.

**Pressure at a point on a horizontal area at a given depth in a liquid under the action of gravity.**—In Fig. 270 is shown an open vessel containing liquid at rest. Consider the equilibrium of a vertical column of the liquid, of height $y$ measured downwards from the free surface. Let the lower end of the column be horizontal and have an area $a$; this area is supposed to be small, and all horizontal sections of the column are taken to be equal and similar.

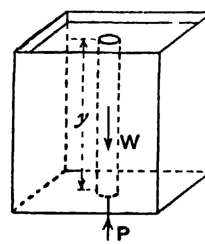


FIG. 270. — Pressure at a given depth in a liquid.

Neglecting any gaseous pressure acting on the free surface, the forces acting on the column are (i) its weight W; (ii) the upward reaction P which the liquid immediately under the foot of the column exerts on the column; (iii) the forces exerted by the liquid surrounding the column; these forces act inwards and prevent the column from spreading outwards. The forces mentioned in (iii) are applied everywhere in directions normal to the vertical sides of the column, and are therefore horizontal. Hence they cannot contribute directly to the equilibrating of the vertical force W. Therefore P and W must be equal, and must act in the same straight line.

If $d$ is the density of the liquid, then $dg$, or $w$, is the weight of the liquid per unit of volume. Let $p$ be the pressure at the depth $y$, then, since the volume of the column is $ay$,

$$\mathsf{P} = \mathsf{W} = way,$$

or,

$$p = \frac{way}{a} = wy. \qquad \text{.............................(1)}$$

It will be noted that $w$ has been assumed to be constant throughout the column, *i.e.* the liquid has been assumed to be incompressible. Hence the result should not be applied to a compressible fluid such as air. Note also that the pressure in a given liquid is proportional to the depth $y$.

**Pressure at a point on an inclined surface.**—In Fig. 271 is shown a vertical column of liquid of rectangular section and having small transverse dimensions $a$ and $b$. AB is the horizontal bottom of the column, and AC is a sloping section. Consider the equilibrium of the wedge ABC, neglecting the weight of the wedge and taking account only of the pressures $p$, $q$ and $r$ acting on AB, BC and CA respectively. As the faces of the wedge are taken very small, we may assume that $p$, $q$ and $r$ are distributed uniformly; hence they give rise to resultant forces $P = p \times AB \times b$, $Q = q \times BC \times b$, and $R = r \times AC \times b$, and these forces act normally at the centres of the faces. Hence P, Q and R intersect at the centre of the circle circumscribing the triangle ABC, and thus comply with one of the conditions of equilibrium of three non-parallel forces. Taking horizontal and vertical components of R, we have

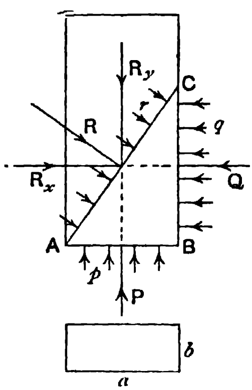


FIG. 271.—Pressure on an inclined surface.

$$R_x = R \sin BAC = Q. \quad \text{.......(1)}$$
$$R_y = R \cos BAC = P. \quad \text{.......(2)}$$

From (1), $\quad r \,.\, AC \,.\, b \,.\, \sin BAC = q \,.\, BC \,.\, b,$

or,
$$r \,.\, \sin BAC = q\frac{BC}{AC} = q \,.\, \sin BAC\,;$$
$$\therefore\ r = q. \quad \text{.......(3)}$$

From (2), $\quad r \,.\, AC \,.\, b \,.\, \cos BAC = p \,.\, AB \,.\, b,$

or,
$$r \cos BAC = p \,.\, \frac{AB}{AC} = p \,.\, \cos BAC\,;$$
$$\therefore\ r = p. \quad \text{.......(4)}$$
$$\therefore\ p = q = r. \quad \text{.......(5)}$$

Strictly speaking, this result is true only when the dimensions of the wedge are reduced indefinitely, in which case the assumptions made become justifiable. In the limit, the wedge becomes a point, lying on three intersecting planes, one horizontal, one vertical, and one inclined, and we may assert that the fluid pressure at the intersection of these planes is the same on each plane, *i.e.* **the pressure at a point in a fluid is the**

**same for any plane passing through that point.** Hence equation (1) (p. 229) becomes available for calculating the pressure at a point on any immersed surface, whatever may be its inclination.

**Head.**—Since the pressure in a given liquid is proportional to the depth $y$ below the free surface, pressures are often measured by stating the value of $y$ and also the name of the liquid; $y$ is then called the **head.** The head may be defined as the vertical height of a column of liquid reaching from the point under consideration up to the free surface level. Thus a head of 30 inches of mercury (density 13·59 grams per cubic cm.) is equivalent to a pressure of 14·7 lb. per square inch, and a head of 144 feet of water (density 62·3 pounds per cubic foot) is equivalent to a pressure of 62·3 lb. per square inch.

Pressures are also sometimes stated in **atmospheres.** One atmosphere may be defined for the present as the pressure produced at the base of a column of mercury 76 centimetres high. This is equivalent to a pressure of $76 \times 13{\cdot}59 = 1032{\cdot}8$ grams weight per square centimetre, or to 1·033 kilograms weight ($= 1{\cdot}0132 \times 10^6$ dynes) per square centimetre. In the British system one atmosphere is taken as the pressure at the base of a column of mercury 30 inches high, and is equivalent to a pressure of 14·7 lb. per square inch.

**The pressure in a liquid at rest is constant at all points in a horizontal plane.**—In Fig. 272 is shown a horizontal row of small liquid cubes, enlarged in the drawing for the sake of clearness. The cubes are actually supposed to be indefinitely small, and may be thus looked upon as forming part of a horizontal line. Let the cube $a$ be at a depth $y$ in the liquid, then the pressure on each of its faces will be $p = wy$ (p. 229). Hence the vertical face in contact with the cube $b$ exerts a pressure equal to $p$ on the vertical face of $b$, and therefore every face of the cube $b$ has a pressure equal to $p$. Similarly, all faces of the cube $c$ will be subjected to pressures equal to $p$, and so on to the end of the row. Thus the pressures at all points in a horizontal immersed line are equal, and since a horizontal line can be drawn in any direction in a horizontal plane, it follows that the pressures at all points in a horizontal immersed plane are equal.

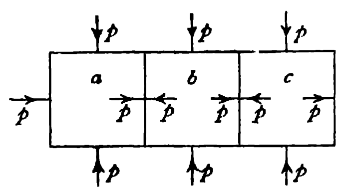



FIG. 272.—Transmission of pressure through a row of cubes.

It follows that the total force which a liquid exerts on a horizontal area may be calculated by taking the product of the pressure and the area.

Let $d$ = the density of the liquid.

$w = dg$ = its weight per unit volum

$A$ = the horizontal area.

$y$ = the depth of the liquid.

Then, Total force $= P = wyA = dg\,yA$. ............................(1)

**The free surface of a liquid at rest is a horizontal plane.**—In Fig. 273, A, B and C are points on an immersed horizontal plane, and are at depths $y_1$, $y_2$ and $y_3$ respectively below the free surface, which we assume not to be a horizontal plane. The pressures at A, B and C are respectively $wy_1$, $wy_2$ and $wy_3$, and these are equal, from what has been said above. Therefore $y_1 = y_2 = y_3$, and hence the free surface must be a plane parallel to that containing A, B and C, and must therefore be a horizontal plane. This result must be modified somewhat for places near the walls of the vessel, where the effect of surface tension causes curvature in the free surface (Chap. XXII).

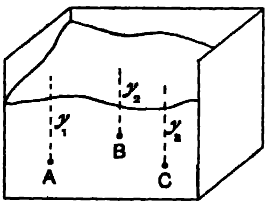


FIG. 273.—Free surface of a liquid at rest.

In Fig. 274 (*a*) is shown a vessel containing liquid at rest, the free surface being AB. Any portion of the liquid, such as CDE, is in equi-

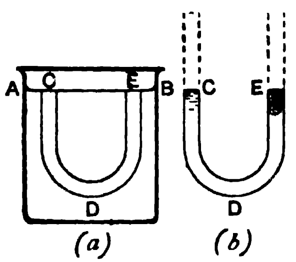


FIG. 274.—Free surfaces in a tube.

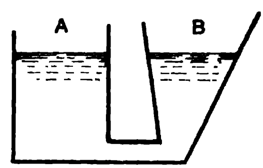


FIG. 275.—Free surfaces in communicating vessels.

librium, and this state will not be disturbed by the enveloping of this portion in a bent tube. The pressures which were supplied initially by the surrounding liquid are now supplied by the walls of the tube. Further, the tube, now full of liquid, may be removed without disturbing the liquid contained in it, *i.e.* the free surfaces at C and E (Fig. 274 (*b*)) will still be in a horizontal plane. If required, both limbs of the U tube may be extended upwards without producing any effect on the state of equilibrium of the liquid. We infer that the free surface of a liquid at rest lies entirely in a horizontal plane, even when the liquid is contained in different but communicating vessels (Fig. 275). This fact leads to the popular statement that water always finds its own level.

EXPT. 41.—**Pressure on a horizontal surface at different depths.** Arrange apparatus as shown in Fig. 276. A is a brass tube suspended vertically from a spring balance C and partially immersed in a liquid contained in a vessel B. The tube is closed at its lower end, and the outside of the bottom

is horizontal. The tube may be loaded internally and may thus be immersed at different depths ; a scale of centimetres engraved on the outside of the tube (zero at the bottom) enables the depth $y$ of the bottom below the free surface to be observed.

It is evident that the total upward force P which the liquid exerts on the bottom together with the upward pull T exerted by the spring balance is equal to the weight W of the tube and contents. Hence

$$P + T = W,$$

or, $$P = W - T.$$

Make a series of experiments, and evaluate P for each ; note the depth $y$ for each experiment. Since the area of the bottom of the tube is constant, P will be proportional to the pressure at the depth $y$. Test if this is so by plotting P and $y$ ; a straight-line graph provides evidence of the truth of the law.

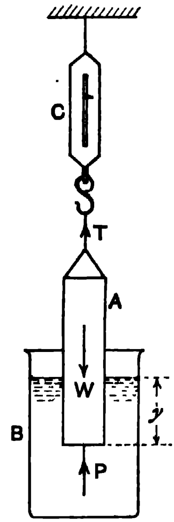



FIG. 276.—Apparatus for finding the pressure at different depths.

**Total force acting on one side of an immersed plate.**—If the plate is horizontal, *e.g.* the horizontal bottom of a vessel containing a liquid, the pressure is uniform and the total force is calculated by taking the product of the pressure and the area.

Let

$w$ = the weight of the liquid per unit volume.

$y$ = the depth of the plate below the free surface.

A = the area of one side of the plate.

Then, Total force exerted on one side = $P = wyA$. ................(1)

The following method is applicable to both vertical and inclined plates (Fig. 277 (*a*) and (*b*)). Let $a$ be a small area of the plate at a

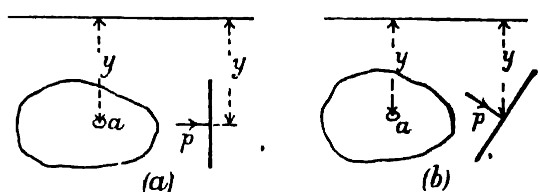



FIG. 277.—Total pressure on immersed surfaces.

depth $y$ below the free surface. Let $p$ be the pressure on $a$ ; then

$$p = wy,$$

and, Force acting on $a = wya$.

This expression applies equally to any other small area of the plate; hence

$$\text{Total force exerted on one side} = \mathsf{P} = w(a_1y_1 + a_2y_2 + a_3y_3 + \text{etc.})$$
$$= w\Sigma ay. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Now $\Sigma ay$ is the simple moment of area of the plate about the free surface of the liquid, and may be calculated by taking the product of the total area A of one side of the plate, and the depth $\bar{y}$ of its centre of area (a point which coincides with the centre of gravity of a thin sheet having the same shape and area as the plate). Thus:

$$\mathsf{P} = w\mathsf{A}\bar{y}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$w\bar{y}$ is the pressure of the liquid at the centre of area; hence the rule: **The total force on one side of an immersed plate is given by the product of the area and the pressure at the centre of area.** Thus the pressure at the centre of area is the average pressure on the plate.

The above proof does not depend upon the surface of the plate being plane, so that the rule applies also to curved surfaces, such as a sphere immersed in a liquid.

EXAMPLE 1.—Find the total force exerted on the wetted surface of a rectangular tank 6 feet by 4 feet by 2 feet deep when full of water.

$$\text{Total force on the bottom} = w\mathsf{A}_1y_1$$
$$= 62{\cdot}3 \times 6 \times 4 \times 2$$
$$= \underline{2990} \text{ lb. weight.}$$
$$\text{Total force on one side} = w\mathsf{A}_2\bar{y}_2$$
$$= 62{\cdot}3 \times 6 \times 2 \times 1$$
$$= \underline{747{\cdot}6} \text{ lb. weight.}$$
$$\text{Total force on one end} = w\mathsf{A}_3\bar{y}_3$$
$$= 62{\cdot}3 \times 4 \times 2 \times 1$$
$$= \underline{498{\cdot}4} \text{ lb. weight.}$$
$$\therefore \text{ Total force on the wetted surface} = 2990 + (747{\cdot}6 + 498{\cdot}4)2$$
$$= \underline{5482} \text{ lb. weight.}$$

EXAMPLE 2.—A cylindrical tank 7 feet in diameter has its circular bottom horizontal and contains water to a depth of 4 feet. Find the total force exerted by the water on the curved wetted surface.

The centre of area of the curved surface lies on the axis of the cylinder at a depth of 2 feet below the free surface; hence

$$\text{Total force on the curved surface} = w\mathsf{A}\bar{y}$$
$$= 62{\cdot}3 \times (\pi d \times 4) \times 2$$
$$= 62{\cdot}3 \times \tfrac{22}{7} \times 7 \times 4 \times 2$$
$$= \underline{10{,}965} \text{ lb. weight.}$$

EXAMPLE 3.—A sphere 8 cm. in diameter is sunk in an oil weighing 0·8 gram per cubic centimetre. The centre of the sphere is at a depth of 40 centimetres. Calculate the total force on the surface of the sphere.

$$\begin{aligned}\text{Total force} &= wA\bar{y}\\ &= 0{\cdot}8 \times 4\pi r^2 \times 40\\ &= 0{\cdot}8 \times 4 \times \tfrac{22}{7} \times 16 \times 40\\ &= \underline{6437} \text{ grams weight.}\end{aligned}$$

The student should note that the total force exerted on the horizontal bottom of a vessel containing a liquid is independent of the shape of the vessel, and consequently is independent of the weight of the contained liquid This follows as a consequence of the pressure being constant over the whole horizontal surface. The total force is $wAy$, and it is evident that this is independent of the shape of the vessel.

**Resultant force exerted by a liquid.**—The total force exerted by a liquid on an area with which it is in contact is the arithmetical sum of the forces which the liquid exerts on the small areas into which the given area may be divided. The resultant force is the vector sum of these forces. In Example 1, p. 234, the total force on the wetted surface of the tank was found to be 5482 lb. weight. It is evident, however, that the total force acting on one side is balanced by the equal total force acting on the opposite side of the tank. Similarly, the total forces acting on the opposite ends balance each other, and therefore the resultant force exerted on the wetted surface is equal to the total force acting on the bottom, viz. 2990 lb. weight.

In the case of all plane surfaces subjected to fluid pressure, the total force and the resultant force are equal. It will also be evident that the resultant force exerted by the liquid contained in a vessel of any shape is equal to the weight of the liquid. This is evident from the consideration that the resultant effect of the reactions of the walls of the vessel is to balance the weight of the contained liquid, and hence the resultant force exerted by the liquid must be equal to this weight, and must act vertically through the centre of gravity of the contained liquid.

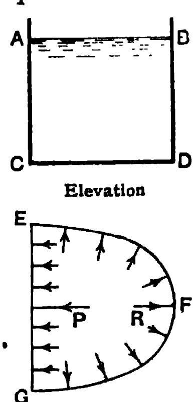



FIG. 278.—Resultant force on plane and curved sides.

In Fig. 278 is shown a vessel having one side plane, vertical and rectangular in shape; this side is EG in the plan and AC in the elevation. The remainder of the sides EFG is vertical, and is curved in the plan. The vessel contains liquid, the free surface of which is AB. That the vessel

is equilibrated horizontally by the liquid pressures is apparent, as may be tested easily by suspending it from a long cord, when no horizontal movement will occur. Hence the resultant force P acting on the plane side EG must be equal and opposite to, and must act in the same straight line as the resultant force on the curved sides. In other words, if components of the forces which act normally on the curved sides be taken in directions perpendicular and parallel to EG, then the arithmetical sum of the components perpendicular to EG will be equal to P. Hence the resultant force R acting on the curved sides may be found by evaluating P.

$$P = wA\bar{y} = w \times (AC \times EG) \times \tfrac{1}{2}AC;$$

$$\therefore\ R = \tfrac{1}{2}w \,.\, AC^2 \,.\, EG.$$

**Centre of pressure.**—The **centre of pressure** of an area exposed to fluid pressure is that point through which the resultant force acts. Let a vertical rectangular area ABDC (Fig. 279) be subjected to the pressure of a liquid, the free surface of which cuts the area in AB. It is evident from symmetry that the centre of pressure G lies in the vertical line HK, which divides the area into two equal and similar parts.

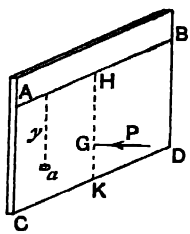


FIG. 279.—Centre of pressure.

To find the depth of G, consider a small area $a$ lying in ABDC and at a depth $y$ below AB. Then

$$\text{Force acting on } a = way.$$

Taking moments about AB, we have

$$\text{Moment of the force acting on } a = way \times y = way^2.$$

This expression serves for the moment of the force acting on any other small portion of the area ABDC; hence the total moment is given by

$$\text{Total moment about AB} = w(a_1y_1^2 + a_2y_2^2 + a_3y_3^2 + \text{etc.})$$
$$= w\Sigma ay^2.$$

$\Sigma ay^2$ is called the **second moment of area**; the form of the expression is similar to that for the moment of inertia of a body, viz. $\Sigma my^2$, and the results given on pp. 177-182 may be used by substituting the total area A for the total mass M. Writing $\Sigma ay^2 = I$, we have

$$\text{Total moment about AB} = wI. \quad \ldots\ldots\ldots\ldots(1)$$

This moment may be expressed in another way. The resultant force P acts through G, therefore

$$\text{Total moment about AB} = P \times GH = wA\bar{y} \times GH, \quad \ldots\ldots\ldots\ldots(2)$$

where $\bar{y}$ is the depth of the centre of area below the free surface. Hence, from (1) and (2),

$$wA\bar{y} \times GH = wI;\quad \therefore\ GH = \frac{I}{A\bar{y}}. \quad \ldots\ldots\ldots\ldots(3)$$

For the rectangular area ABDC (Fig. 279), and for the axis AB,

$$I=\frac{A\times HK^2}{3},\quad \text{and}\quad \bar{y}=\tfrac{1}{2}HK\,;$$

$$\therefore\ GH=\frac{\tfrac{1}{3}A\times HK^2}{A\times\tfrac{1}{2}HK}=\tfrac{2}{3}HK.$$

It will be noted that the position of the centre of pressure is not affected by the kind of liquid, and that $w$ disappears from the final result.

For a vertical circular area touching the free surface (Fig. 280) we have

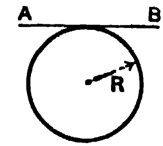


FIG. 280.

$$I=\tfrac{5}{4}AR^2\,;\quad \bar{y}=R\,;$$

$$\therefore\ \text{Depth of the centre of pressure}=\tfrac{5}{4}AR^2/AR$$
$$=\tfrac{5}{4}R.$$

**Pressure diagrams.**—A pressure diagram, for an area subjected to fluid pressure, shows the pressure graphically at all points in the area. The method of construction may be understood by reference to Fig. 281, showing one side of a rectangular tank containing a liquid.

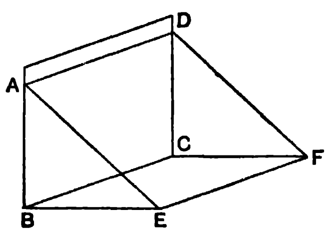


FIG. 281.—Example of a pressure diagram.

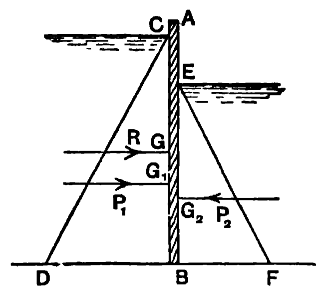


FIG. 282.—A dock gate.

Neglecting the gaseous pressure on the free surface of the liquid, the pressure at A on the side ABCD is zero, and the pressure at B is $w\times AB$. Make $BE=CF=w\times AB$ to any convenient scale of pressure, and join AE, DF and EF. The resulting figure is a wedge, and the pressure at any point in ABCD may be found by drawing a normal at that point to meet the sloping face of the wedge.

EXAMPLE 1.—A gate closing the entrance to a dock is 40 feet wide. There is sea water on one side to a height of 30 feet, and on the other side to a height of 18 feet above the lower edge of the gate. Find the resultant force exerted by the water on the gate.

Referring to Fig. 282 (which is not drawn to scale), AB is the section of the gate, and the pressure diagrams for the high-water and low-water sides

of the gate are CDB and EFB respectively. The total forces on the high-water and low-water sides are $P_1$ and $P_2$ respectively.

$$P_1 = wA_1\bar{y}_1$$
$$= 64 \times (40 \times 30) \times \tfrac{30}{2} = 1{,}152{,}000 \text{ lb. weight.}$$
$$P_2 = wA_2\bar{y}_2$$
$$= 64 \times (40 \times 18) \times \tfrac{18}{2} = 414{,}720 \text{ lb. weight.}$$

$P_1$ acts at the centre of pressure $G_1$, and $BG_1$ is one-third of BC (p. 236) and is therefore 10 feet. Similarly, $P_2$ acts at the centre of pressure $G_2$, and $BG_2$ is $18 \div 3 = 6$ feet. The resultant of $P_1$ and $P_2$ is the resultant force R required in the question.

$$R = P_1 - P_2 = 1{,}152{,}000 - 414{,}720$$
$$= \underline{737{,}280} \text{ lb. weight.}$$

Take moments about B, giving

$$R \times BG = (P_1 \times BG_1) - (P_2 \times BG_2)\,;$$
$$\therefore\ BG = \frac{(1{,}152{,}000 \times 10) - (414{,}720 \times 6)}{737{,}280}$$
$$= \underline{12{\cdot}25} \text{ feet.}$$

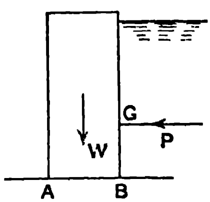


FIG. 283.—A reservoir wall.

EXAMPLE 2.—The wall of a reservoir is rectangular in section (Fig. 283), 9 feet high and 4 feet thick. The free surface of the water is 1 foot below the top of the wall. Take moments about A, and evaluate the ratio,—overthrowing moment of the water/moment of resistance of the weight of the wall. The density of the water is 62·5 pounds per cubic foot, and the density of the material of the wall is 120 pounds per cubic foot.

In examples of this kind it is customary to consider a portion of the wall one foot in length.

$$P = wA\bar{y}$$
$$= 62{\cdot}5 \times (8 \times 1) \times \tfrac{8}{2}$$
$$= 2000 \text{ lb. weight.}$$

And $$BG = \tfrac{8}{3} = 2\tfrac{2}{3} \text{ feet;}$$

$$\therefore \text{ Overthrowing moment} = 2000 \times \tfrac{8}{3}$$
$$= \underline{5333{\cdot}3} \text{ lb.-feet.}$$

$$\text{Weight of the wall} = (9 \times 4 \times 1) \times 120$$
$$= 4320 \text{ lb. weight.}$$

$$\text{Moment of resistance} = 4320 \times \tfrac{4}{2}$$
$$= \underline{8640} \text{ lb.-feet.}$$

$$\therefore \text{ Required ratio} = \frac{5333{\cdot}3}{8640} = \underline{0{\cdot}6173}.$$

## Exercises on Chapter XVIII

**1.** Define a fluid. Distinguish the states solid, fluid and gaseous. Explain why normal stress only may be present in a fluid at rest.

**2.** What is meant by the pressure at a point in a fluid? How is pressure measured? What are the dimensions of pressure?

**3.** Calculate the pressure at a depth of 24·5 cm. in mercury. (Density of mercury = 13·6 grams per cubic centimetre.)

**4.** Find the pressure in lb. weight per square inch at a depth of 2 miles in sea water of density 64 pounds per cubic foot.

**5.** One side of a vessel slopes at an angle of 30° to the vertical. The vessel contains oil having a density of 52 pounds per cubic foot. Find the pressure on the sloping side at depths of 3 and 5 feet.

**6.** Prove that the pressure at a given depth in a liquid is the same on any plane.

**7.** What head of mercury corresponds to a head of 34·6 feet of water? To what pressure is the given head equal? (The density of mercury is 13·6 times that of water.)

**8.** Find the head of water necessary to produce a pressure of one atmosphere. State the result in feet.

**9.** Prove that the free surface of a liquid at rest is a horizontal plane.

**10.** A rectangular tank, 4 feet long, 2 feet broad and 2 feet deep, is full of water (density 62·5 pounds per cubic foot). Find the magnitudes of the total forces on the bottom, on one side and on one end.

**11.** Find the total force acting on the horizontal bottom of a cylindrical tank, 6 feet diameter and 3 feet deep, containing sea water (density 64 pounds per cubic foot) to a depth of 2·75 feet.

**12.** In Question 11 find the total force acting on the curved sides of the tank.

**13.** A tank 10 feet long has a rectangular horizontal bottom 4 feet wide. The ends of the tank are vertical; both sides are inclined at 45° to the horizontal. The tank contains water to a depth of 6 feet. Find the total forces acting on the bottom, on one side and on one end. (Density of water 62·5 pounds per cubic foot.)

**14.** In Questions 10 and 13 find the resultant forces exerted by the liquid on the tanks.

**15.** A hemispherical bowl, 12 cm. in diameter, is full of mercury (density 13·6 grams per cubic centimetre). Find the resultant force exerted by the liquid on the bowl.

**16.** A vessel has the form of an inverted cone, 6 inches diameter of base, 4 inches vertical height, and is full of oil having a density of 51 pounds per cubic foot. Find the resultant force and the total force acting on the curved inner surface of the vessel.

**17.** The ends of a vessel are triangular (Fig. 284). The side AB is vertical, and BC is inclined at 60° to the horizontal. AB = 3 feet, and the length of the vessel is 4 feet. Find the resultant forces acting on the vertical side, on the sloping side and on one end when the tank is full of water (density 62·5 pounds per cubic foot). Find the depth of the centre of pres-

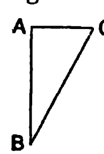


FIG. 284.

sure of the triangular end. (The second moment of area of a triangle about the base is $\frac{1}{6}AH^2$, where A is the area of the triangle and H is its vertical height.)

**18.** A rectangular opening in a reservoir wall is closed by a vertical door 4 feet high and 3 feet wide. The top edge of the door is 20 feet below the surface of the water. Find the resultant force acting on the wetted side of the door ; find also the centre of pressure.

**19.** A hole in the vertical side of a tank containing water is 2 feet in diameter and is closed by a flap. The centre of the hole is 10 feet below the surface of the water. Find the resultant force which the water exerts on the flap, and show where it acts.

**20.** A cylindrical tank is 2 feet in diameter and 3 feet high, and has a vertical partition which divides the tank into two equal compartments: One compartment is full of oil of density 50 pounds per cubic foot, and the other is full of oil of density 55 pounds per cubic foot. Find the resultant forces acting on the inner curved surface of each compartment, and find also the resultant force acting on the partition.

**21.** A reservoir wall is rectangular in section ; the wall is 20 feet long and 7 feet high. The depth of the water is 6 feet. Find the total force which the water exerts on the wall (neglect the pressure of the atmosphere). If the material of the wall weighs 120 lb. per cubic foot, what should be the thickness of the wall in order that the moment of the weight may be twice the overthrowing moment?

**22.** Draw a right-angled triangle ABC ; AB is vertical and is 30 feet high ; BC is horizontal and is 25 feet. The triangle represents the section of a reservoir wall. Take one foot length of the wall and find its weight, if the material weighs 140 lb. per cubic foot. Water pressure acts on the side AB, the free surface being 3 feet below the top of the wall. Find the resultant force which the water exerts on this portion of the wall. Find also the resultant of the force exerted by the water and the weight of the wall ; mark the point in BC through which this force passes, and give its distance from B.

**23.** A dock gate is 12 feet wide. There is fresh water on one side of the gate to a depth of 9 feet, and on the other side to a depth of 6 feet. Find the resultant force which the water exerts on the gate and its position.

**24.** Obtain the dimensions of the units of force, pressure and energy in terms of the units of length, time and mass. Prove that a pressure of a million dynes per square centimetre is equivalent to a pressure of about 15 lb. wt. per square inch, having given that 1 pound = 454 grams, $g = 980$ cm./sec.$^2$ and 1 inch = 2·54 cm. approximately.

**25.** A cubical open vessel of edge 1 ft. is filled with water ; one of the vertical sides is hinged along its upper edge, and can turn freely about it. What force must be applied to the lower edge of the side so as just to keep it from opening? (The weight of a cubic foot of water is $62\frac{1}{2}$ lb.) L.U.

**26.** A sea wall slopes from the bottom at an angle of 30° to the horizon for 20 feet, and is then continued vertically upwards. Find the resultant horizontal and vertical forces on it, in tons weight per yard of its length, when there is a depth of 15 feet of water. (Take a cubic foot of sea water to weigh 64 lb.) L.U.

**27.** A reservoir containing water to a depth of 20 feet has an opening in a vertical side 5 feet wide at the lower edge, 3 feet wide at the upper edge, and 4 feet high, and the lower edge is flush with the bottom of the reservoir. This opening is closed by a plate. If the coefficient of friction between the plate and the side of the reservoir is 0·2, find the force required to move the plate vertically.

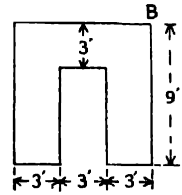


FIG. 285.

**28.** Find the depth of the centre of pressure on a rectangular area immersed in water with one side in the water surface.

The lamina illustrated in Fig. 285 is inserted in water with its plane vertical and with the side AB in the water surface. Find the position of its centre of pressure, proving that it is not actually on the area itself. (Neglect atmospheric pressure.)

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**29.** Prove that the difference of the pressures at two points in a heavy liquid is proportional to the difference of their depths.

A hollow right circular cone of height $h$ and semi-angle $\alpha$ rests with its base on a horizontal table. If the cone is filled with water and the weight of the empty cone is equal to the weight of the water it contains, find the thrust of the water on the base of the cone and the pressure of the cone on the table, explaining briefly why the second result is not double the first.

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# CHAPTER XIX

## HYDROSTATICS (*CONTINUED*). HYDRAULIC MACHINES

**Pressure of the atmosphere.**—The weight of the atmosphere causes it to exert pressure on the surfaces of all bodies. This pressure may be rendered evident by the following experiment.

EXPT. 42.—**Pressure of the atmosphere.** Take a glass tube about 82 cm. in length, sealed at one end and open at the other (Fig. 286). Thoroughly clean and dry the interior of the tube. Fill it with clean mercury. Close the open end with a finger, and invert the tube two or three times so as to collect any contained air into one bubble; allow this bubble to escape and add mercury so as to fill the tube. Close the end with a finger, invert the tube and place its mouth below the surface of mercury contained in a beaker. Withdraw the finger and clamp the tube in a vertical position. It will be found that the mercury level falls to a definite height in the tube. The part of the tube above the mercury contains mercury vapour alone, at a pressure too small to be taken into account. This space is called a **Torricellian vacuum.** The pressure on the surface of the mercury in the tube may thus be taken as zero. At A the pressure of the atmosphere on the free surface of the mercury in the beaker is equal to the pressure inside the tube at the same level. The latter pressure is produced by the weight of the column of mercury in the tube. Let $h$ be the height of the mercury column in centimetres, and let $w$ be the weight of mercury in grams weight per cubic centimetre; then the pressure of the atmosphere at the time of the experiment is

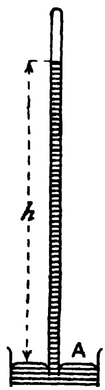


FIG. 286.—Apparatus for showing the principle of the barometer.

$$p = wh \text{ grams weight per sq. cm.}$$

Since $w$ is constant, the height $h$ is used in practice as a measure of the pressure of the atmosphere. The instrument described is a form of **barometer.**

From the observed height of mercury in the barometer, find the pressure of the atmosphere at the time of the experiment in grams weight per square centimetre and also in lb. weight per square inch.

**Effect of gaseous pressure on the free surface of a liquid.**—The pressure of the atmosphere, or other gaseous pressure, on the free surface of a liquid was neglected in Chapter XVIII; it may be taken

into account by the following artifice. In Fig. 287, AB is the free surface of a liquid contained in a vessel and is subjected to a gaseous pressure $p_a$. Let $p_a$ be removed entirely, and let an equivalent pressure be obtained by the addition of another layer of the same liquid. The surface level of the liquid added is CD and is supposed to have no gaseous pressure acting on it. If the weight per unit volume of the liquid is $w$, the depth $y_a$ of the layer may be found from

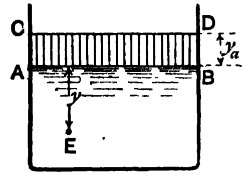


FIG. 287.—Effect of the pressure of the atmosphere.

$$p_a = wy_a,$$

or,

$$y_a = \frac{p_a}{w}. \qquad (1)$$

The pressure $p$ at any point E in the liquid, situated at a depth $y$ below the real free level AB, is given by

$$\begin{aligned} p &= w(y + y_a) \\ &= wy + wy_a \\ &= wy + p_a. \qquad (2) \end{aligned}$$

It may therefore be said that the pressure at any point in the liquid is given by the sum of the pressure due to the weight of the liquid actually in the vessel, and the constant gaseous pressure applied to the free surface. This statement may be generalised by saying: **If, at a given place in a liquid, an additional pressure be applied, then that additional pressure is transmitted unaltered in magnitude to all points in the liquid.**

EXAMPLE.—The vertical side of a rectangular tank is 6 feet long and 4 feet high. If the tank is full of water, find the magnitude of the resultant force acting on the wetted side, taking into account the atmospheric pressure of 15 lb. wt. per square inch.

Due to the pressure of the atmosphere there is a uniform pressure on the wetted side of $15 \times 144 = 2160$ lb. wt. per square foot.

$$\begin{aligned} \text{Total force due to the atmospheric pressure} &= 2160 \times 6 \times 4 \\ &= 51{,}840 \text{ lb. wt.} \end{aligned}$$

Due to the water alone, the total force is given by

$$\begin{aligned} \text{Total force due to the water} &= \text{average pressure} \times \text{area} \\ &= (62{\cdot}3 \times 2) \times (6 \times 4) \\ &= 2990{\cdot}4 \text{ lb. wt.} \end{aligned}$$

The magnitude of the resultant of these forces is given by their sum; hence,

$$\begin{aligned} \text{Resultant force} &= 51{,}840 + 2990 \\ &= \underline{54{,}830} \text{ lb. wt.} \end{aligned}$$

In the case of open vessels and in other similar examples, the pressure of the atmosphere is neglected in practice. It is evident that both the outer and inner surfaces of the sides of the vessel are subjected to equal

pressures by the atmosphere ; hence the resultant forces due to these pressures balance, and the resultant effect on the sides of the vessel is the same as would be experienced by the application to the inner surfaces of the liquid pressures alone.

**Pressure produced by a piston.**—In Fig. 288 (*a*), a vessel A is in communication with a cylinder B, which has a piston C capable of sliding freely in the cylinder, and nicely fitted so as to prevent leakage taking place between the piston and the walls of the cylinder. The vessel A and the portion of the cylinder below the piston are full of liquid. The piston carries a load the weight of which is P, and the area of the piston is $a$ square units. It is evident that the downward force P is balanced by the resultant upward force which the liquid exerts on the piston. The latter force is produced by the pressure of the liquid, and if $p$ be this pressure, we have

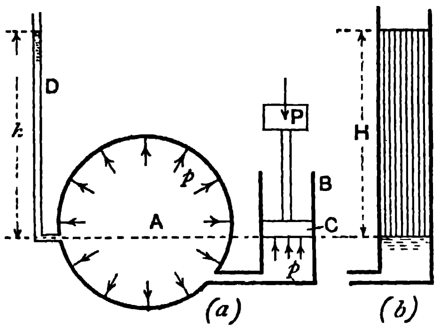


FIG. 288.—Pressure produced by a loaded piston.

$$p = \frac{P}{a}.$$

It is immaterial whether this pressure is produced by means of a loaded piston, as in Fig. 288 (*a*), or by means of a column of liquid, as shown in Fig. 288 (*b*). If H is the head required to produce the pressure $p$, then

$$p = wH,$$

or,

$$H = \frac{p}{w},$$

where $w$ is the weight of the liquid per unit volume.

The pressure $p$ is transmitted uniformly throughout the liquid (p. 243), and hence the inner surfaces of the cylinder, pipes and vessel will be everywhere subjected to this pressure. It will be understood in making this statement that the effects of the weight of the liquid in the vessel are disregarded, and that the effect of the loaded piston alone is being considered. The truth of the above statement may be proved by attaching a glass tube D to the vessel A in Fig. 288 (*a*) at a place on the same level as the lower side of the piston, when it will be found that the liquid rises in the tube to a height $h$, which will be found to be equal to the calculated value of the head H due to $p$. It will be noted that the actual pressure at points above the place where

the tube is connected to A will be less than $p$, and at points below the connection greater than $p$, this being owing to the weight of the liquid in the vessel.

**Hydraulic or Bramah press.**—Very great forces may be obtained by the employment of a liquid under pressure. The principle may be understood by reference to Fig. 289, which shows an outline diagram of a **hydraulic or Bramah press.** A is a cylinder of small diameter fitted with a plunger rod B, which can slide in the cylinder. A load P is applied to B, thus producing pressure in the liquid which fills the lower part of the cylinder. A pipe E connects B with another cylinder C, having a diameter considerably larger than that of B. C is fitted with a ram D, which can slide in the cylinder C. The ram carries a load W. Since the pressure of the liquid is uniform throughout, we may calculate the relation of W and P as follows:

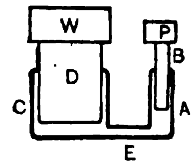

FIG. 289.—Principle of the hydraulic press.

Let $d$ = the diameter of the plunger B.

D = „ „ „ ram D.

$p$ = the pressure of the liquid.

Then, $$p\frac{\pi d^2}{4} = \mathrm{P}; \quad \text{and} \quad p\frac{\pi \mathrm{D}^2}{4} = \mathrm{W};$$

$$\therefore \frac{\mathrm{W}}{\mathrm{P}} = p\frac{\pi \mathrm{D}^2}{4} \Big/ p\frac{\pi d^2}{4} = \frac{\mathrm{D}^2}{d^2}. \quad \ldots\ldots\ldots(1)$$

It will be noted that the effect of friction in preventing free movement of the plunger and ram in the cylinders has been neglected in the above.

So far we have considered only the static balancing of W and P; the arrangement, however, becomes a machine if we permit the plunger B to descend. Liquid is then forced out of the cylinder A and must find accommodation in the cylinder C; therefore the ram D and load W must rise. If B descends a distance H while D rises a distance $h$, P does PH units of work while W$h$ units of work are done on W. Neglecting frictional waste, we have by the principle of the conservation of energy,

$$\mathrm{PH} = \mathrm{W}h,$$

or, $$\frac{\mathrm{H}}{h} = \frac{\mathrm{W}}{\mathrm{P}} = \frac{\mathrm{D}^2}{d^2}, \quad \ldots\ldots\ldots(2)$$

an expression which gives the velocity ratio of the machine.

The principle of the hydraulic press is used in many hydraulic machines. The liquid generally employed is water. The cylinder A in Fig. 289 represents a **hydraulic pump**, which in practice is so arranged

as to deliver a constant stream of water under high pressure to the cylinder C.

**Transmission of energy by a liquid under pressure.**—In the hydraulic press discussed above it is apparent that the load P gives up potential energy while descending, and at the same time the load W is acquiring potential energy. Thus energy has been transmitted from one place to another by the medium of the flow of liquid under pressure. It is evident that the transmission of energy will continue so long as P is allowed to descend, *i.e.* so long as flow is kept up in the liquid under pressure. This principle is made use of in hydraulic power installations. Water is brought to a high pressure by means of pumps in a central station, and the water is led through pipes to various points in the district at which energy is required, and where machines capable of utilising this energy are installed.

**Pressure energy of a liquid.**—In Fig. 290, AB is a pipe having a piston C capable of sliding along the pipe. Liquid under a pressure $p$ enters the pipe at A, and work is done in forcing the piston in the direction from A towards B against a resistance R. Let the area of the piston be $a$ square units, and let the piston move through a distance L. Then the resultant force P acting on the left-hand side of the piston is equal to $pa$, and the work done is given by

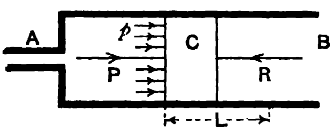


FIG. 290.—Pressure energy of a liquid.

$$\text{Work done by P} = \text{PL} = pa\text{L}. \qquad \text{.......................(1)}$$

It is evident that the volume described by the piston is $a$L, and this is equal to the volume V of liquid which must be admitted at A in order to keep the pipe full of liquid while the piston is moving. Hence

$$\text{Work done by P} = p\text{V}. \qquad \text{..........................(2)}$$

As this work has been done by supplying a volume V of liquid, we have,

$$\text{Work done per unit volume} = p. \qquad \text{....................(3)}$$

Suppose the water to be supplied from an overhead cistern situated at a height $h$ above A, and that the density of the liquid is $d$. The weight of the liquid per unit volume is $dg$, therefore

$$p = dgh.$$

Hence we may say,

$$\text{Work done by expending a mass } d \text{ of liquid} = p = dgh.$$

$$\text{Work done by expending unit mass of liquid} = \frac{p}{d} \qquad \text{.............(4)}$$

$$= gh. \qquad \text{...........(5)}$$

The **pressure energy** of a liquid is defined as the energy which can be

derived by expending unit mass of the liquid in the manner described; hence,

$$\text{Pressure energy} = \frac{p}{d} = gh. \quad \text{.........................(6)}$$

Absolute units of force have been employed in the above discussion; hence the quantities involved in (4), (5) and (6) must be expressed as follows:

| | C.G.S. | British |
|---|---|---|
| $p$ | dynes per sq. cm. | poundals per sq. foot. |
| $d$ | grams per c.c. | pounds per cubic foot. |
| $h$ | centimetres. | feet. |
| $g$ | centimetres per sec. per sec. | feet per sec. per sec. |
| Pressure energy | ergs (*i.e.* centimetre-dynes) per gram of liquid. | foot-poundals per pound of liquid. |

Example 1.—Water is supplied by a hydraulic power company at a pressure of 700 lb. wt. per square inch. How much pressure energy in foot-lb. is available per pound of water?

$$\text{Pressure} = p = 700 \times 144g = 100{,}800g \text{ poundals per sq. foot.}$$

$$\text{Pressure energy} = \frac{p}{d} = \frac{100{,}800g}{62{\cdot}3} \text{ foot-poundals}$$

$$= \frac{100{,}800}{62{\cdot}3} = \underline{1618} \text{ foot-lb. per pound of water.}$$

Example 2.—Some mercury is under a head of 30 cm. of mercury. What is the pressure energy?

$$\text{Pressure energy} = gh = 981 \times 30 = \underline{29{,}430} \text{ ergs per gram of mercury.}$$

**Hydraulic transmission of energy.**—The principal apparatus required in a hydraulic installation is shown in outline in Fig. 291. A is a

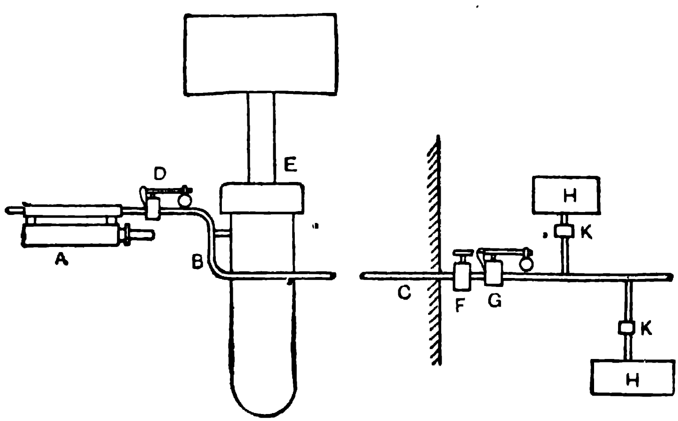


Fig. 291.—Diagram of a hydraulic installation

**hydraulic pump** driven by a steam engine, or other source of power, and delivers water under high pressure into the pipe system BC. A **safety valve** is provided at D and permits some of the water to escape should the pressure become dangerously high. Near to the pump is situated a **hydraulic accumulator** E, which is connected to the pipe system, and maintains constant pressure in the water. A branch pipe from the main pipe system is led into the consumer's premises, and a stop valve F enables him to cut the supply off when necessary. There is also a safety valve G, which serves to protect his machinery from damage due to any excessive pressure. The machines H, H are operated by the water; each machine has a valve K, by use of which the machine may be started and stopped.

A typical **hydraulic pump** is shown in Fig. 292. A cylinder A is fitted with a piston B, which may be pushed to and fro by means of a rod C operated by an engine. Hydraulic packing at D renders water-tight the hole through which the rod passes. The valves F and K are discs which rise and fall vertically, thus opening and closing passages E and L through which the water may pass. The piston is shown moving towards the right, and water is flowing into the cylinder from E past the open valve F. At the same time, the water on the right-hand side of the piston is being expelled under high pressure through a passage G into H and so into the delivery pipe.

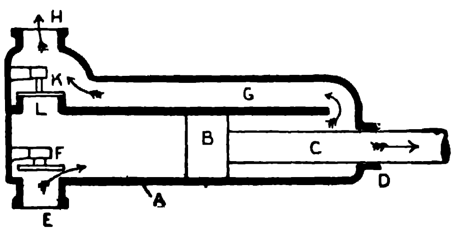

FIG. 292.—Section of a hydraulic pump.

When the piston is moving towards the left, the valve F drops and closes E. The water on the left-hand side of the piston is then forced under high pressure through L, past the valve K (which has now lifted), and so partly into the delivery pipes at H, and partly through the passage G into the right-hand side of the cylinder. The pump thus delivers water during each stroke of the piston.

A **hydraulic accumulator** is illustrated in Fig. 293, and consists of a cylinder fitted with a ram which passes through a hole at the top of the cylinder and carries a load W on the top. The cylinder is connected to the pipe system of the plant (Fig. 291), and therefore the ram is subjected to the same pressure as that in the pipes.

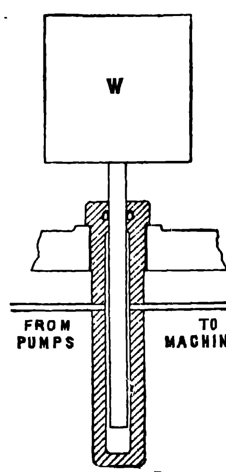

FIG. 293.—Diagram of a hydraulic accumulator.

Let $p$ = the pressure of the water, lb. wt. per sq. in.

$d$ = the diameter of the ram, inches.

Then, Resultant upward force on the ram $= p \times \pi d^2/4 = W$ lb. wt.

Since $d$ is constant, it is apparent that the working pressure depends on the magnitude of W, which accordingly determines the maximum pressure which may exist in the pipes.

The accumulator has another very important function. Suppose that all the machines operated by the water are cut off and that the hydraulic pumps continue to work. Owing to the incompressibility of water, either some of the pipes would be burst or the whole of the energy expended in giving pressure to the water would be wasted in the flow through the safety valve. The accumulator prevents both damage and waste. Under the conditions mentioned, the water delivered by the pumps causes the ram of the accumulator to rise. If H be the height through which the ram travels, the load W stores potential energy to the amount WH, which is available for doing useful work when the machines are started again.

A system of levers, not shown in Fig. 293, is operated by W when the accumulator has been raised to the maximum safe height; the lever system is connected to the engine driving the pumps and cuts off the steam, thus stopping the pumps. Directly the machines are started again, the ram begins to descend and the lever system is operated in the reverse direction, thus restarting the pumps. The whole arrangement is automatic, and the pumps in the power house start and stop in answer to any demand for water from premises situated perhaps a considerable distance away.

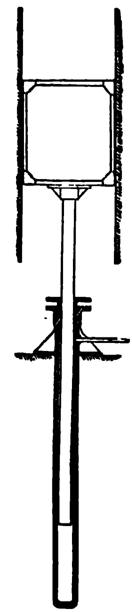

FIG. 294.—A direct-acting hydraulic lift.

**Hydraulic lift.**—A simple type of **hydraulic lift** is shown in Fig. 294, and consists of a hydraulic cylinder fitted with a ram which carries a cage on its top. The total weight of the ram, cage and load carried in the cage must be equal to the resultant force which the water exerts on the ram, neglecting friction.

**Hydraulic engine.**—Fig. 295 shows a common form of hydraulic engine whereby the pressure energy possessed by water under pressure may be converted into useful work. The engine has three cylinders A, B and C arranged at angles of 120°, each fitted with a piston—that at A is shown in section. Each piston is connected by a rod to a crank DE, which is fixed to a shaft capable of rotating about D. The water

acts on the outer sides of the pistons only, and is admitted and discharged by an arrangement of valves not shown in Fig. 295.

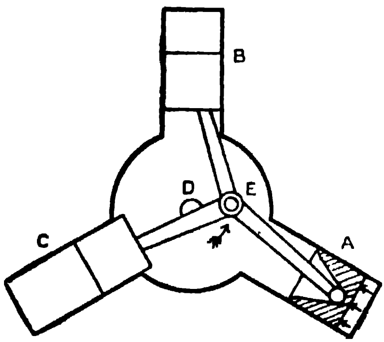


FIG. 295.—Three-cylinder hydraulic engine.

The piston in A has just commenced to move towards D, and is doing work on the crank; that at C is just finishing its movement towards D, and the piston in B is moving away from D; the water in the latter cylinder is flowing out of the cylinder, and is finished with so far as the derivation of energy is concerned. Thus there is always at least one piston which is doing work on the crank, and continuous rotation of the shaft D is secured.

The horse-power may be calculated in the following manner:

Let $p$ = the water pressure in lb. wt. per square inch.
$d$ = the diameter of each cylinder in inches.
L = the travel, or stroke of the piston towards D in feet.
N = the revolutions per minute of the shaft.

Then,

Resultant force exerted by the water on one piston $= p \times \pi d^2/4$ lb. wt.

Work done on one piston per stroke $= p\pi d^2/4 \times \mathrm{L}$ foot-lb.

As there are three pistons, there will be 3N strokes per minute, during each of which work will be done; hence

$$\text{Work done per minute} = p \times \pi d^2/4 \times \mathrm{L} \times 3\mathrm{N} \text{ foot-lb.}$$

And,
$$\text{Horse-power} = \frac{3p\pi d^2\mathrm{LN}}{4 \times 33{,}000}.$$

**Pressure of a gas.**—In dealing with a gas such as air, the pressure may be measured above absolute zero of pressure. Absolute zero of pressure may be defined as the state of pressure in a closed vessel containing no substance in the gaseous state, and this empty space is termed a **perfect vacuum.** Pressures measured from a perfect vacuum are called **absolute** pressures.

In practical work, the pressure of a gas is measured by an appliance called a **pressure gauge,** several types of which are described in the Part of the volume on Heat. Pressure gauges indicate the difference between the existing absolute pressure of the atmosphere and the absolute pressure inside the closed vessel containing the gas. The pressure of the atmosphere is denoted by zero on the graduated scale

of the gauge, and other pressures are measured as so much above, or below, the pressure of the atmosphere; hence the term **gauge pressure.** Consider a closed vessel containing a gas under high pressure. If the absolute pressure of the gas is $p$, and the absolute pressure of the atmosphere is $p_a$, then the pressure indicated by the pressure gauge is $(p - p_a)$, and we have

Absolute pressure = gauge pressure + pressure of the atmosphere.

**Boyle's law.**—Experiments on the relation of the pressure and volume of gases will be described later. These show that, for gases such as air, hydrogen, oxygen and nitrogen under ordinary conditions of pressure and temperature, the absolute pressure is inversely proportional to the volume, provided the temperature is kept constant. Taking a given mass of gas, we have

$$p \propto \frac{1}{v},$$

or, $$pv = \text{a constant.}$$

If the initial conditions of pressure and volume are $p_1$ and $v_1$, and if the final conditions are $p_2$ and $v_2$, then

$$p_1 v_1 = p_2 v_2.$$

This law was discovered by Boyle and bears his name.

**Lift pumps.**—The **lift pump** depends for its action on the pressure exerted by the atmosphere. In Fig. 296, A is a cylinder fitted with a piston or pump bucket B; this piston has a valve which opens upwards, thus permitting water to pass from the lower to the upper side through holes in the piston. The cylinder is connected by a pipe C, having a foot valve D at its bottom, to a cistern of water E. The pump is operated by means of a rod which is attached to the bucket and passes through a hole in the top cover of A.

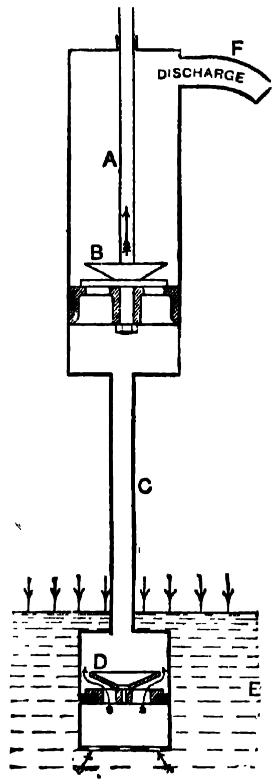


FIG. 296.—Section of a lift pump.

During the up-stroke of the bucket, the valve B is closed and D is open; the pressure of the air in C falls, and the pressure of the atmosphere on the surface of the water in E causes some water to flow up the pipe. During the down-stroke, the valve D closes and B opens. No water can pass D now, and some air will be expelled through B. Repetition of

these operations will bring water ultimately into the cylinder A, when it will pass B and be discharged through F. The process of starting in this manner is long, and may be hastened by first charging the cylinder and pipe C with water.

Taking the pressure of the atmosphere to be equivalent to a head of 30 inches of mercury, or $30 \times 13{\cdot}59 = 407{\cdot}7$ inches of water, we see that the pressure of the atmosphere is incapable of forcing water to a height greater than about 34 feet. The cylinder of a lift pump is placed usually at a height not exceeding 30 feet from the free surface of the water in the well.

**Force pumps.**—In **force pumps** the piston is employed for forcing liquids into vessels in which the pressure is higher than that of the atmosphere. For example, the pump employed to feed water into a steam boiler has to force the water to enter the boiler against the pressure of the steam in the boiler. Such a pump is shown in section in Fig. 297. A is the cylinder with a ram, or plunger, B. Water enters the cylinder, passing the valve C, during the upward stroke of the plunger, and is delivered through another valve D during the downward stroke of the plunger. The valve D opens when the pressure in the cylinder A, produced by forcing the plunger downwards, becomes greater than that exerted on the upper side of the valve.

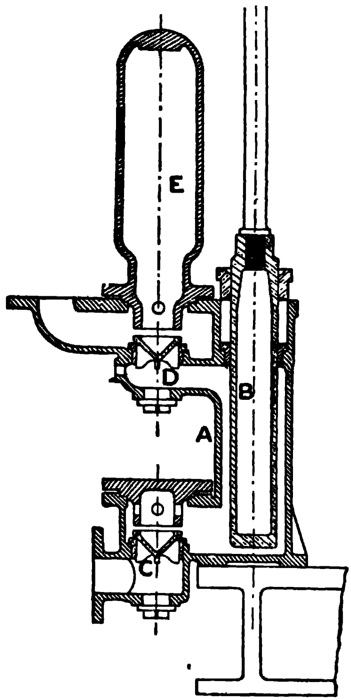

FIG. 297.—Section of a boiler feed pump.

This pump is fitted with an air-vessel E, the action of which is of interest. The vessel is in communication with the discharge pipe of the pump, and is closed entirely otherwise. Air is contained in the upper part of the vessel, and is compressed, during the early part of the downward stroke of the plunger, by some of the water discharged from A entering the vessel. Water being practically incompressible, absence of the soft cushion provided by the air in the air-vessel would lead to shocks due to the action of the plunger when it meets the water during the downward stroke, and might possibly cause the pipes to

burst. Further, the pump shown in Fig. 297 is single-acting, *i.e.* water is delivered during the downward stroke only. During the upward idle stroke of the plunger, the compressed air in the air-vessel maintains some flow of water along the discharge pipe, and thus assists in producing a continuous pumping action.

## Exercises on Chapter XIX

**1.** If the mercury in a barometer falls from 29·8 to 29·4 inches, find the difference in the total forces which the atmosphere exerts on the outer surface of a sphere 2 feet in diameter.

**2.** An open rectangular tank is 6 feet long, 4 feet wide and 3 feet deep, and is full of fresh water. Find the total forces on the interior surfaces of the bottom, one side and one end, taking account of the pressure of the atmosphere of 15 lb. wt. per sq. in.

**3.** In a hydraulic or Bramah press the ram is 15 inches in diameter and the pump plunger is 2 inches in diameter. What is the velocity ratio of the machine? If the pressure of the water is 1000 lb. wt. per square inch, what force must be applied to the pump plunger, and what force will be exerted by the ram? Neglect friction.

**4.** What is the pressure energy of water when under a pressure of 1200 lb. wt. per square inch? State the result in foot-lb. per pound mass of water.

**5.** How many gallons of water, under the conditions given in Question 4, must be supplied per hour in order to maintain a rate of working of one horse-power? (There are 10 pounds of water in one gallon.)

**6.** Water at a pressure of 700 lb. wt. per square inch acts on a piston 1 square foot in area and the piston has a stroke of 1 foot. How much work is done (*a*) by the total volume of water admitted, (*b*) by one pound of the water? If the water company charges 20 pence per thousand gallons of water, how much energy is given for each penny?

**7.** A vertical tube, 3 metres high, is full of mercury. What is the pressure energy per gram of the mercury at the bottom of the tube?

**8.** The load of a hydraulic accumulator is 130 tons weight, and the ram is 20 inches in diameter. Find the pressure of the water in lb. wt. per square inch.

**9.** The ram of a hydraulic accumulator is 7 inches in diameter, and the stroke is 12 feet. If the pressure of the water is 700 lb. wt. per square inch, find the weight of the load. How much water is stored when the ram is at the top of the stroke? Find also the energy then stored.

**10.** In the simple form of goods lift shown in Fig. 294, the ram is 3 inches in diameter and has a stroke of 12 feet. If the water is supplied under a pressure of 700 lb. wt. per square inch, what total load can be raised, neglecting friction? How much work is done in raising this load?

**11.** The hydraulic engine shown in Fig. 295 has three rams, each 3·5 inches in diameter and having a stroke of 6 inches. The pressure of the water supplied is 120 lb. per square inch, and the engine runs at 90 revolutions per minute. Neglect waste, and find the horse-power. If the efficiency is 65 per cent., find the useful horse-power.

**12.** The pressure in a closed vessel is known to be 150 lb. wt. per square inch above that of the atmosphere. The barometer reads 29·6 inches of mercury. Find the absolute pressure inside the vessel.

**13.** If the volume of a given mass of gas is 450 cubic centimetres when the absolute pressure is 2000 cm. of mercury, find the volume if the absolute pressure falls to 550 cm. of mercury without change in temperature.

**14.** A vertical tube has its lower end immersed in a bath of mercury, and an air pump is connected to the upper end of the tube. The barometer stands at 30 inches of mercury. By means of the pump the pressure in the interior of the tube is lowered to 10 lb. wt. per square inch absolute. Find the height at which the mercury in the tube will stand above that in the bath.

**15.** In a lift pump (Fig. 296) the pump bucket is 14 inches in diameter, and has a stroke of 2 feet. If the pump makes 20 double strokes (one upwards and one downwards) per minute, how many cubic feet of water will be raised per hour, neglecting waste?

**16.** In Question 15 the moving parts of the pump (bucket, rod, etc.) weigh 150 lb., and the level of the water in the well is 15 feet below the top of the discharge pipe. What total upward force must be applied to the pump rod when the bucket is ascending? Neglect friction.

**17.** A lift pump is used to pump oil of specific gravity 0·8 from a lower into an upper tank. What is the maximum possible height of the pump above the lower tank when the pressure of the atmosphere is 30 inches of mercury?

**18.** A boiler feed pump (Fig. 297) is single-acting, and the plunger has a stroke of 12 inches. The pump makes 60 double strokes per minute, and has to force 20,000 pounds of water per hour into a boiler working at a pressure of 160 lb. wt. per square inch. Neglect waste and friction, and find (*a*) the diameter of the plunger, (*b*) the force which must be applied to the plunger during the downward stroke.

# CHAPTER XX

## FLOATING BODIES. SPECIFIC GRAVITY

**Resultant force exerted by a liquid on a floating or immersed body.**—In Fig. 298 (*a*) is shown a body floating at rest, in still liquid. Equilibrium is preserved by the action of two forces, viz. the weight W acting vertically through the centre of gravity of the body, and the resultant force R exerted by the liquid. It is evident that these forces must act in the same vertical line, and that they must be equal and of opposite sense. The force R is called the **buoyancy**.

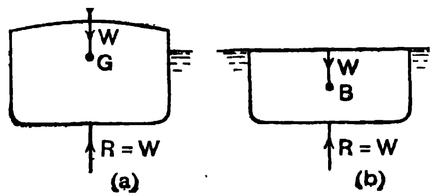

FIG. 298.—Equilibrium of a floating body.

Imagine for a moment that the liquid surrounding the body becomes solid, and so can preserve its shape; let the body be removed, leaving a cavity which it fits exactly (Fig. 298 (*b*)). Let this cavity be filled with some of the same liquid, and let the surrounding liquid resume its ordinary state. The pressures on the liquid now filling the cavity are identical with those which formerly acted on the body, and the effect is the same—the weight of the liquid is balanced. Hence the weight of the liquid filling the cavity and the weight of the body must be equal, since each is equal to R, the resultant force exerted by the surrounding liquid.

Further, in Fig. 298 (*b*), R must act through the centre of gravity of the liquid filling the cavity; this centre is called the **centre of buoyancy**. It is clear that, since R acts in the same vertical line in both figures, the centre of buoyancy B, and the centre of gravity of the body G, must fall in the same vertical. Hence we have the statement: **When a body is floating at rest in still liquid, the weight of the body is equal to the weight of the liquid displaced by the body, and the centres of gravity of the body and of the displaced liquid are in the same vertical line.**

A little consideration will show that the same method of reasoning applies also to a body totally immersed in a liquid and that the same result follows. Thus, the upward resultant force, or buoyancy, which water exerts on a piece of lead lying at the bottom of a tank is equal to the weight of the water displaced by the lead.

The **principle of Archimedes** follows from the above facts, viz.: **A body wholly, or partially, immersed in a liquid experiences an apparent loss of weight which is exactly equal to the weight of the liquid displaced.**

**Stability of a floating body.**—The state of equilibrium of a body floating at rest in still liquid may be determined by slightly inclining the body (Fig. 299); the originally vertical line passing through G, the centre of gravity of the body, now occupies the position XY. The weight W of the body acts through G, and the resultant force R exerted on the body by the liquid acts vertically upwards through the centre of buoyancy B. It will be noted, since more liquid is now displaced on the right-hand side of XY, that the tendency has been to move B a little to the right of its first position while the body was being inclined. In Fig. 299, R and W form a couple tending to restore the body to its original position; hence the equilibrium is stable. Produce R upwards, cutting XY in M; M is called the **metacentre**. If M falls above G, as in Fig. 299, the equilibrium is stable. If M coincides with G (as in the case of a rubber ball floating on water), the lines of R and W coincide and the equilibrium is neutral. If M falls below G, the couple will have the sense of rotation opposite to that shown in Fig. 299, and has an upsetting tendency; the equilibrium was therefore unstable. The determination by calculation of the position of the metacentre is beyond the scope of this book.

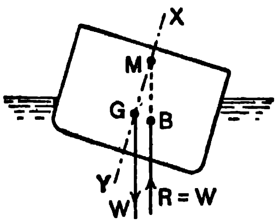


FIG. 299.—Stability of floating body.

**Force required to equilibrate an immersed body.**—Should the weight of an immersed body be exactly equal to that of the liquid displaced by the body, then the forces of weight and buoyancy balance one another, and the body is in equilibrium. Otherwise, an upward or downward force must be applied to the body, depending on whether the weight of the body is greater or smaller than that of the liquid displaced. In Fig. 300 (*a*), the weight of the body is greater than the buoyancy B, hence an upward force P is required to maintain equilibrium. In Fig. 300 (*b*), B is greater than W, and a downward force P is required in order to ensure total immersion.

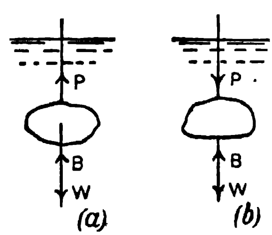


FIG. 300.—Equilibrium of immersed bodies.

In Fig. 300 (*a*), $P + B = W$.

In Fig. 300 (*b*), $P + W = B$.

A **pontoon** is a closed or partially open vessel used sometimes for raising sunken wrecks from the bottom in water of moderate depth.

In Fig. 301, two pontoons, A and B, support a stage CD, having hoisting tackle at E and F. Chains are placed round the sunken body G, which may thus be raised from the bottom. The total pull in the chains is equal to the weight of the sunken body diminished by the weight of the liquid displaced by the body.

In **floating docks** (Fig. 302) a large vessel A, forming the dock, may be sunk to the position shown at (*a*) by the artifice of admitting water

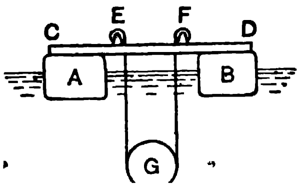


FIG. 301.—Use of pontoons.

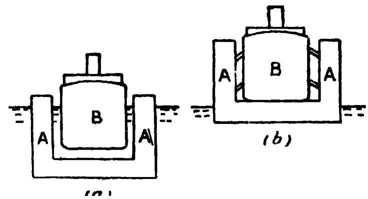


FIG. 302.—A floating dock.

into internal tanks. The ship B may then float into the dock. On pumping the water out of the tanks, the dock rises slowly out of the water, and the ship rests on the floor. Ultimately the position shown in Fig. 302 (*b*) is attained, in which the ship is entirely out of the water.


FIG. 303.—A submarine boat.

The immersion of **submarine boats** is similarly accomplished by means of internal water tanks. When cruising, the free surface level is AB (Fig. 303), and a considerable portion of the boat is above water. The vessel may be sunk lower in the water by admitting water into internal tanks; the free surface may then be at CD, or even higher. Pumps are provided in the interior for emptying the water tanks, and thus bringing the vessel again to its original level. When the boat is in motion, diving may be accomplished by the use of horizontal rudders, which cause the longitudinal axis of the boat to become inclined.

**Specific gravity.**—The **specific gravity** of the material of a given body is defined as the ratio of the weight of the body to the weight of an equal volume of water. In Great Britain the comparison is made generally at 60° F., or 15° C.

Let $W_s$ = the weight of the body,

$W_w$ = the weight of an equal volume of water, both expressed in the same units.

Then, $$\text{Specific gravity} = \rho = \frac{W_s}{W_w} \quad \text{.........................(1)}$$

The weight of any body may be calculated from a knowledge of its volume V and specific gravity $\rho$. Thus, if $w_0$ be the weight of unit

volume of water, the weight of the body, if made of water, is $Vw_0$, and the actual weight is

$$W = Vw_0\rho. \quad \text{..............................(2)}$$

**Relation between the density and specific gravity of a given substance.**—It will be remembered (p. 2) that the density of a substance is its mass per unit volume.

Let $M$ = the mass of a body.
$V$ = its volume.
$d$ = the density of the material.
$\rho$ = the specific gravity of the material.
$w_0$ = the weight of unit volume of water, in absolute units.
$W$ = the weight of the body, in absolute units.

Then,
$$W = Mg = Vdg = Vw_0\rho.$$
$$\therefore \frac{d}{\rho} = \frac{w_0}{g} = \text{a constant.} \quad \text{..........................(3)}$$

In the C.G.S. system, $w_0$ is the weight of a cubic centimetre of water and is $g$ dynes; hence in this system the same number expresses both the specific gravity and the density of a given substance. In the British system, $w_0$ is the weight of a cubic foot of water; taking the density of water at 60° F. to be 62·3 pounds per cubic foot, $w_0$ is $62{\cdot}3g$ poundals; hence in this system

$$d = 62{\cdot}3\rho.$$

It follows from these relations that any experiment having for its object the determination of the specific gravity of a substance at 60° F. gives also the density of the substance at the same temperature.

EXPT. 43.—**Determination of the specific gravity of a liquid by weighing equal volumes of the liquid and of water.** A specific gravity or density bottle is employed (Fig. 304), and is a small glass bottle having a fine stem. The bottle is filled with liquid by warming it slightly and dipping its mouth into the liquid; repetition of this process will ultimately fill the bottle. The bottle and its contents are then brought to the temperature of 60° F. approximately by standing the bottle for some time in a beaker of water maintained at 60° F.

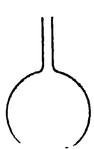

FIG. 304.—Specific gravity bottle.

Weigh the empty bottle; let this be $W_1$ grams weight. Fill the bottle with distilled water, taking care to get rid of any air. Bring the contents to the temperature required, and if necessary add some more water in order to fill the bottle completely. Weigh again, and by subtracting $W_1$ from the result, find $W_w$ grams weight, *i.e.* the weight of the water alone. Empty the bottle and dry it thoroughly. Fill it again, with the liquid under test, adjust the temperature and again weigh. From the result

deduct $W_1$, thus giving $W_s$ grams weight for the weight of the liquid alone. Now $W_s$ and $W_w$ occupied equal volumes; hence

$$\rho = \frac{W_s}{W_w}.$$

$$d = \rho \text{ grams per cubic centimetre}$$
$$= 62{\cdot}3\rho \text{ pounds per cubic foot.}$$

EXPT. 44.—**Specific gravity of a solid by weighing in air and in water.** First weigh the solid in air; let the result be $W_1$ grams weight. Arrange a balance and a vessel of water as shown in Fig. 305, and suspend the solid by means of a fine thread attached to one arm of the balance. The solid should be completely immersed in the water, the temperature of which should be adjusted as nearly as possible to 60° F., or 15° C. Weigh again, thus determining the pull $W_2$ grams weight in the thread. If the buoyancy is $B$ gram's weight, then

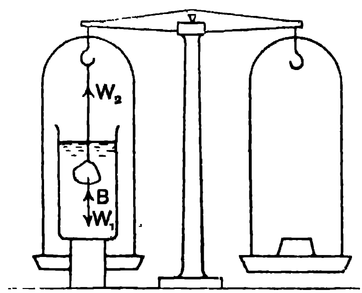


FIG. 305.—Weighing a body in water.

$$W_2 + B = W_1;$$
$$\therefore B = W_1 - W_2.$$

Now $B$ is the weight of a quantity of water having a volume equal to that of the solid; hence

$$\rho = \frac{W_1}{B} = \frac{W_1}{W_1 - W_2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

In this way determine the specific gravities of the samples of iron, brass, lead, etc., supplied.

If some liquid other than water had been employed in the second weighing operation, let the specific gravity of this liquid be $\rho'$. Then

$$\text{Weight of the liquid displaced} = W_1 - W_2.$$

$$\text{Weight of an equal volume of water} = \frac{W_1 - W_2}{\rho'}.$$

$$\text{Specific gravity of the solid} = \frac{W_1\rho'}{W_1 - W_2}. \quad \ldots\ldots\ldots\ldots(2)$$

EXPT. 45.—**Specific gravity of a liquid by weighing a solid in it.**—Use the apparatus shown in Fig. 305. Weigh the solid (*a*) in air, (*b*) in water, (*c*) in the liquid. Let the results be $W_1$, $W_2$ and $W_3$ grams weight respectively. Then,

$$\text{Weight of the water displaced by the solid} = W_1 - W_2.$$
$$\text{Weight of the liquid displaced by the solid} = W_1 - W_3.$$

The volumes occupied by the water and liquid displaced are equal; hence

$$\text{Specific gravity of the liquid} = \frac{W_1 - W_3}{W_1 - W_2}.$$

You are supplied with a piece of brass and some turpentine. Find the specific gravity of the turpentine.

**Variable immersion hydrometer.**—A **hydrometer** is an instrument which can float in the liquid to be tested and by means of which the specific gravity of the liquid may be determined. The instrument shown in Fig. 306 consists of a glass bulb weighted with some mercury contained in an enlargement at the bottom of the bulb, for the purpose of making the instrument float in an upright position. A graduated glass stem is attached to the bulb. Since the weight of the instrument is constant, and is equal always to the weight of the liquid displaced, it follows that the free surface of the liquid will cut a division on the stem depending on the specific gravity of the liquid. Deeper immersion will occur with lighter liquids. British instruments generally have the stems calibrated for a temperature of 60° F., and the liquid to be tested should be brought to this temperature. Variable immersion hydrometers can be used for a limited range only, and therefore a number of instruments is required if there is considerable difference in the specific gravities of the liquids to be tested.

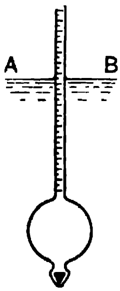


FIG. 306.—Variable immersion hydrometer.

EXPT. 46.—**Calibration of variable immersion hydrometers.** Obtain a glass tube with a bulb blown at one end (Fig. 306), but with the upper end of the tube open. Place small lead shot in the tube until it floats upright with the surface level of the water about 2 cm. above the bulb. Make a mark on the tube at the water level and call this mark X. Take out the tube, dry it and weigh it. Let the mass be $M_X$ gm. Now add lead shot until the tube floats in water to about 2 cm. from its upper end. Again mark the water level and call this Y, and find the weight of the whole tube and shot as before. Let it be $M_Y$ gm.

Assuming the density of water to be unity, the volume of the bulb and stem to X is $M_X$ c.c., and the volume to Y is $M_Y$ c.c.; and if $l$ cm. is the length of tube between X and Y, the area of cross-section of the tube is $\frac{M_Y - M_X}{l}$ sq. cm. Suppose that the tube is now placed in a liquid of density $\delta$, and floats with the liquid surface $x$ cm. below Y. Then the volume immersed is $\left(M_Y - x\frac{M_Y - M_X}{l}\right)$ c.c. But the mass of the tube and shot is $M_Y$, so that, by the principle of Archimedes, the mass of liquid displaced is also $M_Y$, and its density is therefore $\dfrac{M_Y}{M_Y - x\dfrac{M_Y - M_X}{l}} = \delta$ gm. cm.$^{-3}$. Since

$M_X$, $M_Y$ and $l$ are known, a graph connecting $x$ and $\delta$ may be plotted, and the values of $x$ for equal steps in $\delta$ may be found. Then by placing marks on the tube at the appropriate values of $x$, the hydrometer will indicate directly on the scale the density of the liquid in which it is floating.

Note that in the above case the liquid has a density greater than unity. If it is desired to use the hydrometer for liquids of less density than water, it is necessary to load it and find the point Y first. Then remove shot and find the point X, and use the relation $\dfrac{M_X}{M_X + x\dfrac{M_Y - M_X}{l}} = \delta$, $x$ being now measured from X.

EXPT. 47.—**Specific gravity of a solid by use of Nicholson's hydrometer.** This instrument is shown in Fig. 307, and is a hydrometer of constant immersion. A hollow metal vessel C is loaded so as to float upright, and has a wire stem D, which carries a scale-pan E. Another scale-pan is attached at F. A scratch on the stem D determines the standard depth of immersion, and the instrument must be loaded so that the free surface AB cuts this mark.

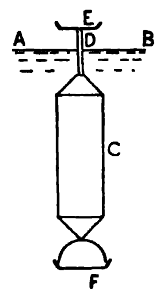

FIG. 307.—Nicholson's hydrometer.

Float the instrument in water, and ascertain what weight $W_1$ grams must be placed in E in order to bring the instrument to standard immersion. Remove $W_1$, and place the body under test into the scale-pan E; add weights $W_2$ to E so as again to produce standard immersion. Then

Weight of the body in air

$$= W = W_1 - W_2 \text{ grams weight. } \qquad (1)$$

Now place the body in the scale-pan F (use a fine thread to tie it down if the body is lighter than water); place weights $W_3$ grams in the scale-pan E in order to secure standard immersion. Then

$$\text{Weight of the body in water} = W' = W_1 - W_3 \text{ grams weight. } \qquad (2)$$

The difference of (1) and (2) gives the weight of the water displaced by the body; hence

$$\begin{aligned}\text{Weight of the water displaced} &= W - W' \\ &= (W_1 - W_2) - (W_1 - W_3) \\ &= W_3 - W_2 \text{ grams weight. } \qquad (3)\end{aligned}$$

$$\therefore \rho = \frac{W_1 - W_2}{W_3 - W_2}. \qquad (4)$$

In this way determine the specific gravities of the various samples supplied.

EXPT. 48.—**Relative specific gravities of liquids which do not mix.** The U tube shown in Fig. 308 contains two liquids which do not mix; one liquid occupies the tube lying between A and B, and the other occupies the portion BC; the surface of separation is at B. Let the specific gravities of these liquids be $\rho_1$ and $\rho_2$ respectively. Let D be at the same level as B; then the pressure at D is equal to the pressure at B, *i.e.*

$$w_1h_1 = w_2h_2,$$

or,
$$\frac{w_1}{w_2} = \frac{\rho_1}{\rho_2} = \frac{h_2}{h_1}.$$

Measure $h_2$ and $h_1$, and evaluate the ratio of the specific gravities. If the specific gravity of one of the liquids is known, find the specific gravity of the other liquid.

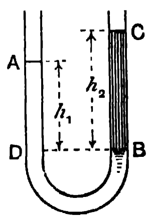


FIG. 308.—Apparatus for liquids which do not mix.

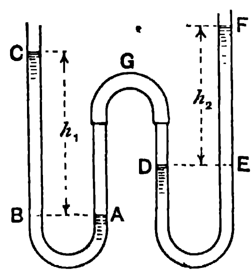


FIG. 309.—Apparatus for liquids which mix.

EXPT. 49.—**Relative specific gravities of liquids which mix.** Fig. 309 shows two U tubes connected by a short rubber tube at G. One liquid occupies the space ABC, and the other occupies the space DEF. The air trapped in AGD prevents contact of the liquids, and exerts the same pressure on the surfaces at A and D. Therefore

$$p_A = p_D.$$

A and B are at the same level, as are also D and E.

$$\therefore\ p_A = p_B = p_D = p_E;$$
$$\therefore\ w_1h_1 = w_2h_2;$$
$$\therefore\ \frac{w_1}{w_2} = \frac{\rho_1}{\rho_2} = \frac{h_2}{h_1}.$$

Measure $h_1$ and $h_2$, and evaluate the ratio.

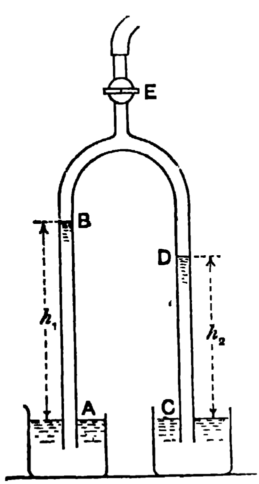


FIG. 310.—Inverted U tube for liquids which mix.

EXPT. 50.—**Relative specific gravities of two liquids which mix, by inverted U tube.** In Fig. 310 an inverted U tube is shown having a branch at the top furnished with a stop-cock, and connected to an air pump by means of which air may be withdrawn from the tube. The lower open ends of the tube are immersed in two

liquids contained in separate vessels. On operating the pump, the superior pressure of the atmosphere on the surfaces of the liquids in the vessels will cause the liquids to rise in the tubes. The pressures inside the tubes at A and C are equal to that of the atmosphere; also the air in the upper part of the tube exerts equal pressures on the surfaces at B and D. Hence

$$w_1h_1 = w_2h_2 ;$$

$$\therefore \frac{w_1}{w_2} = \frac{\rho_1}{\rho_2} = \frac{h_2}{h_1}.$$

Measure $h_1$ and $h_2$, and evaluate the ratio.

**Specific gravity of mixtures of liquids.**—We will suppose that the volume and specific gravity of each liquid are known, that no chemical action occurs, and that there is no change in the volumes. The total volume V c.c. after mixing will be equal to the sum of the volumes $V_1$, $V_2$, $V_3$, etc., in c.c., of the separate liquids; further, there will be no change in weight during mixing; hence the weight W grams after mixing is equal to the sum of the initial weights $W_1$, $W_2$, $W_3$, etc.

$$V = V_1 + V_2 + V_3 + \text{etc. c.c.} \quad \ldots\ldots\ldots\ldots(1)$$

$$W = W_1 + W_2 + W_3 + \text{etc. grams weight.} \quad \ldots\ldots\ldots\ldots(2)$$

From (2), $\quad V\rho = V_1\rho_1 + V_2\rho_2 + V_3\rho_3 + \text{etc.},$

where $\rho$ is the specific gravity of the mixture and $\rho_1$, $\rho_2$, etc., are the specific gravities of the separate liquids; hence

$$\rho = \frac{W}{V} = \frac{V_1\rho_1 + V_2\rho_2 + V_3\rho_3 + \text{etc.}}{V_1 + V_2 + V_3 + \text{etc.}}. \quad \ldots\ldots\ldots\ldots(3)$$

If the weight and specific gravity of each liquid are known, then we proceed as follows:

$$V_1 = \frac{W_1}{\rho_1}; \quad V_2 = \frac{W_2}{\rho_2}; \quad V_3 = \frac{W_3}{\rho_3}, \text{ etc.};$$

$$\therefore V = \frac{W_1}{\rho_1} + \frac{W_2}{\rho_2} + \frac{W_3}{\rho_3} + \text{etc.};$$

$$\therefore \rho = \frac{W}{V} = \frac{W_1 + W_2 + W_3 + \text{etc.}}{\dfrac{W_1}{\rho_1} + \dfrac{W_2}{\rho_2} + \dfrac{W_3}{\rho_3} + \text{etc.}}. \quad \ldots\ldots\ldots\ldots(4)$$

If the volume changes during mixing, becoming V′ say, then, since the weight after mixing is equal to the sum of the weights before mixing, we have

$$V'\rho = V_1\rho_1 + V_2\rho_2 + V_3\rho_3 + \text{etc.};$$

$$\therefore \rho = \frac{V_1\rho_1 + V_2\rho_2 + V_3\rho_3 + \text{etc.}}{V'}. \quad \ldots\ldots\ldots\ldots(5)$$

## Exercises on Chapter XX

**1.** A ship displaces a volume of 400,000 cubic feet of fresh water. Find the weight of the ship. If the ship sails into sea water (64 lb. weight per cubic foot), what volume of water will it displace?

**2.** A rectangular pontoon is required to carry a load of 4 tons weight, and the depression when the load is applied is not to exceed 6 inches in fresh water. Find the horizontal area of the pontoon in square feet.

**3.** A closed cylindrical vessel is 6 feet in diameter and 15 feet long, and weighs 5000 lb. If the vessel is floating in fresh water with the axis of the cylinder in the plane of the water surface, what load is it carrying?

**4.** A body weighing 8 lb. and having a volume of 15 cubic inches lies at the bottom of a tank of fresh water. What force does it exert on the bottom of the tank?

**5.** A piece of iron weighing 4 lb. is immersed in oil weighing 50 lb. per cubic foot, and is supported by means of a cord to which it is tied. If the iron weighs 0·26 lb. per cubic inch, what is the pull in the cord?

**6.** Some oil is poured into a vessel containing some water. Describe what will happen if the liquids do not mix. Give reasons.

**7.** A rectangular body weighing 3·3 lb. in air has dimensions as follows: 4·2 inches long, 2·4 inches wide, 1·1 inches thick. What is the specific gravity of the material?

**8.** A piece of lead, specific gravity 11·4, weighs 0·32 lb. in air. What will be the apparent loss of weight when the lead is immersed in water?

**9.** The density of steel is 480 pounds per cubic foot. What is the specific gravity? Explain why the density and specific gravity of a substance are represented by the same number in the C.G.S. system.

**10.** A plank of wood measures 6 feet by 9 inches by 3 inches, and the specific gravity is 0·6. How many cubic inches will be below the surface if the plank is floating at rest in fresh water? What vertical force must be applied in order to immerse the plank?

**11.** A piece of zinc weighs 42 grams in air, and 37·8 grams when immersed in oil having a specific gravity of 0·7. Find the specific gravity of the zinc.

**12.** A piece of brass weighs 2 lb. in air, and the specific gravity is 8·5. Find the pull in the suspending cord when the brass is immersed in a liquid having a specific gravity of 0·82.

**13.** The weight of a submarine boat is 200 tons, and it lies damaged and full of water at the bottom of the sea. If the specific gravity of the material is 7·8, find the total pull which must be exerted by the lifting chains in order to raise the vessel from the bottom. Take the specific gravity of sea water = 1·025.

**14.** A piece of brass was found to weigh 22·68 grams in air and 20·04 grams in water, and was then used as a sinker for determining the specific gravity of a piece of cork. The cork weighed 1·595 grams in air, and sinker and cork together weighed 14·275 grams in water. Find the specific gravities (*a*) of the brass, (*b*) of the cork.

**15.** To determine the length of a given tangle of copper wire the following measurements were made: Diameter (measured by means of a screw gauge), 0·0762 cm.; weight of the tangle in air, 5·43 grams; and in water, 4·81 grams. Find the specific gravity of the copper and the length of the wire.

**16.** The specific gravity of a piece of brass was found by use of a Nicholson's hydrometer, and the following observations were recorded: Weight required to sink the hydrometer to the standard mark, 4·48 grams; with the brass in the upper pan, 2·22 grams weight were required in the upper pan; with the brass in the lower pan, 2·48 grams weight were required in the upper pan. Find the specific gravity of the brass.

**17.** Some water is introduced into a U tube and fills about 12 cm. of each vertical limb. Some oil of specific gravity 0·8 is poured into one limb and fills a length of 6 cm. of the tube. Find the difference in levels of the free surfaces of the water and oil. (No mixing takes place.)

**18.** An inverted U tube (Fig. 310) has the open end of one limb immersed in water and the other open end is immersed in a liquid having a specific gravity 0·85. The air pump is then worked until the water stands in the tube to a height of 20 cm. Find the height to which the liquid will rise in the other tube.

**19.** Three liquids, A, B, C, are mixed and no chemical action takes place. The volumes and specific gravities are as follows:

| Liquid | A | B | C |
|---|---|---|---|
| Volume, c.c. | 20 | 16 | 24 |
| Specific gravity | 0·88 | 0·76 | 0·92 |

If no change in volume occurs, find the specific gravity of the mixture. If the volume after mixing is 59·6 c.c., find the specific gravity.

**20.** A closed box, whose external dimensions are 3 ft. by 2 ft. by 1 ft., is made of iron of specific gravity 7·8; show that the greatest thickness of the iron, supposed uniform, consistent with the box floating in water without sinking, is 0·42 inch nearly.

**21.** State the *principle of Archimedes.*

Lead shot is added to a uniform test-tube until it floats upright in a liquid of density 1·05 gm. cm.$^{-3}$ If the tube has cross-section 1·5 sq. cm. and it sinks to a depth of 8 cm., what is the weight of the tube and shot?

To what depth would it sink in a liquid of density 0·9 gm. cm.$^{-3}$?

**22.** Explain how Nicholson's hydrometer may be used to find the specific gravity (*a*) of a liquid, (*b*) of a solid heavier than water. Give a sketch of the instrument.

**23.** A U tube, whose ends are open, whose section is one square inch, and whose vertical branches rise to a height of 33 inches, contains mercury in both branches to a height of 6·8 inches. Find the greatest amount of water that can be poured into one of the branches, assuming the specific gravity of mercury to be 13·6. L.U.

**24.** Explain how you would compare the specific gravities of two liquids that mix by means of a U tube.

**25.** Three ingots of equal weights consist of gold, silver, and a gold-silver alloy respectively. If the gold loses weight 21 gm. when immersed in water, the silver 39 gm., and the alloy 27 gm., what proportion by weight of the alloy consists of gold? What is the density of the alloy? (Density of silver = 10·5 gm. per c.c.) C.W.B., H.C.

**26.** State and prove Archimedes' Principle. One end of a uniform straight rod 240 cm. long and weighing 20 kgm. is held by a vertical string. The rod is inclined to the horizontal so that 80 cm. of the other end is

immersed in water. Find the specific gravity of the material of the rod and the tension in the string. C.W.B., H.C.

**27.** Distinguish between mass and weight. Describe in detail an accurate form of balance and explain how it is used to determine the mass of a body. What factors determine the sensitiveness of such a balance?

The apparent mass of a body, whose volume is one litre, is 50 grams when determined by an ordinary chemical balance and brass weights. Find the true mass of the body assuming the density of air to be 1·3 grams per litre and that of brass to be 8·4 grams per c.c. L.U.Hi.Sch.

**28.** A solid cubical block of side $a$, made of uniform material of specific gravity $s$ ($<1$), is held just above the water surface in a large vessel, the lower horizontal face of the cube just touching the water surface. Prove that when the cube sinks down so that a depth $x$ is immersed, the potential energy of the water and the block increases by $W(x^2 - 2xas)/2as$, where $W$ is the weight of the block. Show that the equilibrium position of the block when it is floating freely corresponds to a minimum value of this potential energy. C.W.B., H.C.

**29.** A man can float in sea water with only part of his body immersed. In fresh water an upward force of 4 lb. wt. is required to keep him afloat with the same amount immersed. Find the man's weight, assuming that a cubic foot of sea water weighs 64 lb. and a cubic foot of fresh water 62 5 lb. If, when the supporting force of 4 lb. wt. is removed, the man still floats, find what additional volume is immersed. J.M.B., H.S.C.

**30.** State the principle of Archimedes. A thin rectangular board of specific gravity $s$ is hinged along its shorter side to the flat bottom of a tank. Find the position assumed by the board when water is poured in to a depth $h$; and prove that the board assumes the vertical position when $h$ becomes equal to $\sqrt{s}$ times the length of the board. J.M.B., H.S.C.

# CHAPTER XXI

## LIQUIDS IN MOTION

**Steady and unsteady motion in fluids.**—The motion of a fluid may be either **steady** or **unsteady**. In steady motion, each particle in the fluid travels in precisely the same path as the particle preceding it, thus setting up stream lines or filaments, which may be either straight or curved. Thus, if a fine jet of coloured water be injected into a mass of water moving with steady motion, the coloured water will pursue the stream line which passes through the point of injection and will move unbroken, giving a coloured band which appears to remain fixed in position, and may be curved or straight, depending on the conditions under which flow takes place.

In unsteady or **turbulent motion**, eddies are formed in the fluid. If a coloured jet be injected into water moving with unsteady motion, no colour band is formed; the jet breaks up at once, and colours faintly and uniformly a considerable portion of the water.

Osborne Reynolds used the colour band method to demonstrate steady and unsteady flow of water along a glass pipe. At low speeds of flow, a fine jet of coloured water, introduced into the body of water entering the pipe at one end, travels unbroken along the pipe and indicates steady flow (Fig. 311 (*a*)). As the speed of flow is increased, a critical velocity is reached, above which the colour band breaks up and mingles with the whole of the water in the pipe, thus indicating unsteady flow (Fig. 311 (*b*)).

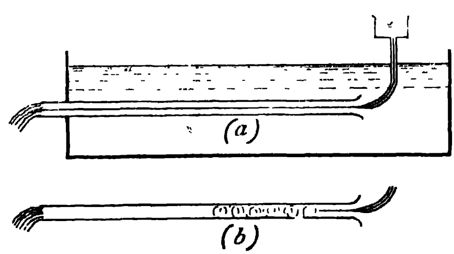


FIG. 311.—Steady and unsteady motion.

Before turbulence sets in, the term **stream line** is applied to the motion of the fluid. In this case the resistance to motion follows definite laws (p. 276), and the pressure gradient required to produce motion can in many cases be calculated. This pressure gradient, however, increases very rapidly when turbulence occurs, much of the applied energy being wasted in producing turbulence. Hence the advantage gained in designing the outline of bodies moving through a fluid so that turbulence is avoided. Such shapes are seen in the design of aeroplanes,

motor-cars, etc., which avoid producing turbulence in the air through which they pass, and so cause the fluid motion to follow stream lines.

Osborne Reynolds found the critical velocity separating turbulence from stream-line flow to be about $1000\frac{\eta}{\rho a}$, where $\eta$ is the coefficient of viscosity (p. 276), $\rho$ the density of the liquid, and $a$ the radius of the tube, all in C.G.S. units.

**Pressure on stream lines.**—Since force is required to change the direction of motion of a body, it follows that straight stream lines can exist only provided there is no resultant force acting on the boundary of the stream line in a direction perpendicular to that of the motion of the fluid (p. 195). In other words, the pressures which the adjacent stream lines exert on the filament under consideration must be uniformly distributed all over the boundary of the filament.

In a mass of fluid moving in curved stream lines, the concave side of any stream line is in contact with the convex side of an adjacent stream line (Fig. 312 (*a*)). The pressure which the concave side *ab* of the lower stream line exerts on the convex side *ab* of the upper stream line is equal and opposite to the pressure which the convex side of the upper stream line exerts on the concave side of the lower stream line. Let this pressure be $p$. The pressure on the concave side *cd* will be less than $p$ by a small amount $\delta p$, and that on *ef* will be greater than $p$ by another small amount $\delta p$. Applying the same reasoning to all stream lines in a body of fluid moving steadily in a curved path (Fig. 312 (*b*)), we see that the pressure $p_1$ on the convex boundary *ab* will diminish gradually across the stream, attaining a lower value $p_2$ at the concave boundary *cd*.

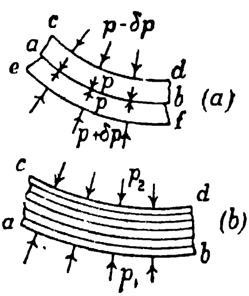

FIG. 312.—Transverse pressures on curved stream lines.

**Total energy of a liquid.**—The total energy at any point in a liquid in motion may be separated into three different kinds of energy, and is expressed conveniently as so much energy per unit mass of liquid: (*a*) **Potential energy,** due to the elevation $h$ above some arbitrary datum level, and given by $gh$ absolute units of energy per unit mass. (*b*) **Pressure energy,** due to the pressure $p$, in absolute units, at the point under consideration, and given by $p/d$ absolute units of energy per unit mass, $d$ being the density of the liquid (p. 246). (*c*) **Kinetic energy,** due to the motion of the liquid, and given by $v^2/2$ absolute units of energy per unit mass, $v$ being the speed of the liquid at the point under consideration. These energies are mutually convertible, *i.e.* any one kind may be converted into either of the other two kinds of energy. The

total energy of the liquid at the point in question is obtained by taking the sum. Thus,

$$\text{Total energy} = gh + \frac{p}{d} + \frac{v^2}{2} \cdot \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

**Bernoulli's theorem.**—Suppose that a small portion of liquid flows from one point to another point, and that the change of position is effected without incurring any waste of energy, then, from the principle of the conservation of energy, we may assert that the total energy is not changed during the displacement. This statement is known as **Bernoulli's theorem,** and leads to the following equation.

Let $h_1$, $p_1$ and $v_1$ be respectively the elevation, pressure and velocity at a certain point in a liquid having a mass $d$ per unit volume, and let the liquid flowing past this point arrive at another point where the elevation, pressure and velocity are respectively $h_2$, $p_2$ and $v_2$. Then

$$gh_1 + \frac{p_1}{d} + \frac{v_1{}^2}{2} = gh_2 + \frac{p_2}{d} + \frac{v_2{}^2}{2} \cdot \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

**An illustration of Bernoulli's theorem.** In Fig. 313, AC is a glass tube having a contraction at B ; a branch D is attached at the middle of the contraction and dips into coloured water in a vessel E. The tube ABC is arranged horizontally, and is connected to a water-tap by means of rubber tubing attached to A. On opening the tap, water flows through the tube and is discharged into the atmosphere at C. Notice also that the coloured water in E ascends D, mingles with the water flowing along ABC and is also discharged at C. The arrangement constitutes a kind of lift pump, and has been used in a modified form for raising water.

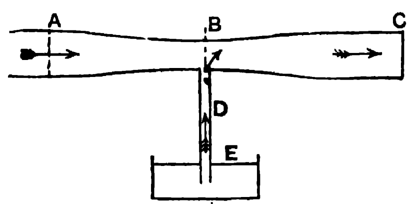

FIG. 313.—Apparatus for illustrating Bernoulli's theorem.

The action may be explained as follows : The tube ABC being horizontal, there will be no change in the potential energy of the water flowing along the tube. Imagine the branch D to be closed for a moment, then, neglecting any wasted energy, the sums of the pressure and kinetic energies at A, B and C will be equal. It is also evident, since the tube ABC is everywhere full of water, that the same quantity of water per second passes every cross-section of the tube ; therefore the velocity at B must be greater than the velocity at C. Hence the kinetic energy at B is greater than the kinetic energy at C, and therefore the pressure energy at B must be less than the pressure energy at C. Now the pressure at C is equal to that of the atmosphere, therefore the pressure at B must be less than that of the atmosphere. Hence, if the branch D be opened, the pressure of the atmo-

sphere on the free surface of the water in E will cause this water to ascend D, and, provided the branch is not too long, to join the water flowing along ABC.

**The siphon.**—The siphon consists of a bent tube, usually made with the limbs of unequal length, and is employed for emptying liquid from a vessel without the necessity for tipping the vessel.

On filling a vessel with water (Fig. 314) the free surface is AB. If both limbs of the siphon CDEF are filled with water and the ends closed by applying the fingers, the siphon inverted and placed in the position shown in Fig. 314 and the fingers removed, it will be found that water is discharged at F until the level in the vessel falls to C.

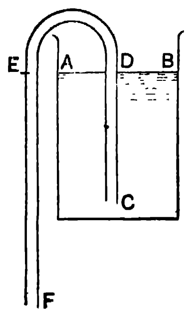

FIG. 314.—Use of a siphon.

The action may be explained as follows: Suppose the flow to be stopped by applying a finger at F; the pressure at D inside the tube would then be equal to the pressure of the atmosphere acting on AB. If the flow be started again, the water at D has taken up some kinetic energy, and has therefore parted with an equivalent amount of pressure energy, and its pressure is now less than that of the atmosphere. Hence there is a resultant effort tending to produce motion from AB downwards through the vessel and thence through CD towards D.

Consider now the column EF, E being on the same level as AB. On its upper surface and acting downwards there is a pressure equal to that at D; the pressure of the atmosphere acts upwards at F, and the weight of the column of liquid acts downwards. There is thus a net downward tendency which causes the liquid to be discharged at F.

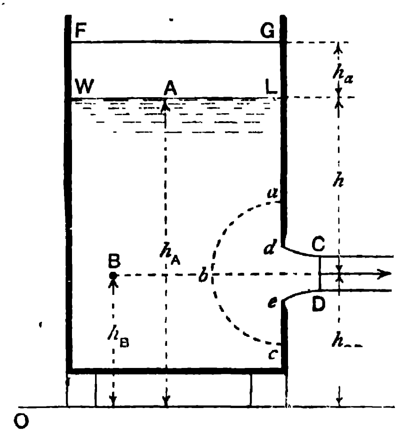

FIG. 315.—Discharge through a sharp-edged orifice.

**Discharge from a sharp-edged orifice.**—Bernoulli's theorem may be applied to the flow of water or other liquid through a small sharp-edged circular orifice. Reference is made to Fig. 315, in which *de* is the orifice; OX is an arbitrary datum level. The free surface level of the liquid is at WL, and is maintained at a constant height *h* above the centre of the orifice by allowing liquid to flow into the tank at a rate equal to that of the discharge through the orifice.

The pressure of the atmosphere, $p_a$ in absolute units, is taken into account by removing the gaseous pressure from WL and substituting a layer of liquid FGLW, having a depth $h_a$; the imaginary free surface FG has then no gaseous pressure acting on it. If the density of the liquid is $d$, then

$$p_a = dgh_a; \qquad h_a = \frac{p_a}{dg}. \tag{1}$$

At A the liquid has potential energy due to the elevation $h_A$; its pressure energy is due to the pressure of the atmosphere $p_a$; the velocity is too small to be taken into account, and the kinetic energy is assumed to be zero

At B the liquid has potential energy due to the elevation $h_B$; the pressure energy is due partly to the head $h$ and partly to the pressure $p_a$ transmitted through the liquid; the velocity, and hence the kinetic energy, is again assumed to be zero.

As the liquid approaches the orifice, its velocity begins to be important on crossing an imaginary boundary $abc$ (Fig. 315). B$b$ being taken as a horizontal line passing through the centre of the orifice, the liquid particles at B pass along the straight line B$b$ and are discharged; other particles, such as those at $a$ and $c$, have to pass round the edges of the orifice, and can do so only by pursuing curved paths. Hence the jet contracts after passing the plane of the orifice $de$. The exterior surfaces of the jet are subjected to the pressure of the atmosphere $p_a$, but the interior of the jet has pressures in excess of $p_a$ up to the section CD, where contraction is complete. After passing CD, the pressure throughout the interior of the jet is equal to $p_a$.

At CD the liquid has potential energy due to the elevation $h_{CD}$; its pressure energy is due to $p_a$; the kinetic energy is due to the velocity $v$ (Fig. 315).

Consider unit mass of liquid initially at A, then passing slowly downwards to B, and thence along B$b$ until it acquires the velocity $v$. The energies above stated may be tabulated in absolute units as follows:

| | A | B | CD |
|---|---|---|---|
| Potential energy - - | $gh_A$ | $gh_B$ | $gh_{CD}$ |
| Pressure energy - - | $\frac{p_a}{d} = gh_a$ | $g(h + h_a)$ | $gh_a$ |
| Kinetic energy - - | 0 | 0 | $\frac{1}{2}v^2$ |

By Bernoulli's law, if there is no waste of energy during the passage of the liquid, the total energies at each of the three places are equal. Hence

$$gh_A + gh_a + 0 = gh_B + g(h + h_a) + 0 = gh_{CD} + gh_a + \tfrac{1}{2}v^2. \quad \text{.........(2)}$$

The following results may now be written:

$$g(h_A - h_B) = gh. \quad \text{................................(3)}$$

This result simply verifies the fact that the gain in pressure energy in passing from A to B (Fig. 315) is equal to the potential energy transformed in descending through the height $h$. Also $h_B$ and $h_{CD}$ are equal, therefore

$$\tfrac{1}{2}v^2 = gh. \quad \text{.................................(4)}$$

This result indicates that the kinetic energy gained is also equal to the potential energy given up in descending through the height $h$. From (4),

$$v^2 = 2gh, \quad \text{.................................(5)}$$

and we may therefore state that the velocity of the jet is the same as would be acquired by a body falling freely from the free surface level to the level of the centre of the orifice.

Experiment shows that the area of the cross-section of CD is about 0·64 of the area of the circular orifice $de$ (Fig. 315). Also the actual velocity at CD is about 0·97 of the velocity given by (5).

Let A = the area of the circular orifice.

Q = the actual quantity of liquid discharged per second.

Then,

$$Q = 0{\cdot}64A \times 0{\cdot}97\sqrt{2gh}$$
$$= 0{\cdot}62A\sqrt{2gh}. \quad \text{.................................(6)}$$

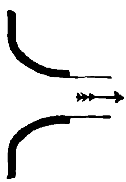

FIG. 316.—A trumpet orifice.

Contraction may be eliminated by use of a **trumpet orifice** (Fig. 316), in which case the area of the jet will be equal to that of the orifice. The velocity is about $0{\cdot}96\sqrt{2gh}$, and

$$Q = 0{\cdot}96A\sqrt{2gh}. \quad \text{..................(7)}$$

**Water-wheels.**—There are large natural stores of energy in the water contained in lakes elevated above the level of the sea. The utilisation of this energy has provided many interesting problems for engineers. The old-fashioned method was to employ a **water-wheel.** A suitable place was selected on a river or stream where there was either a natural waterfall, or where an artificial fall could be obtained by building a dam across the stream. A difference in level being thus obtained, the water was led to the water-wheel, of which there are three types.

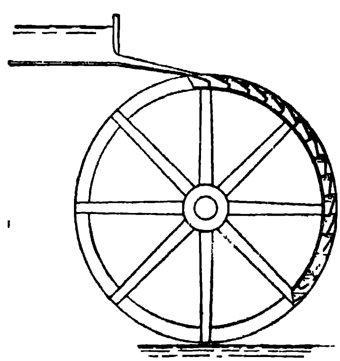

FIG. 317.—Over-shot water-wheel.

In the **over-shot wheel** (Fig. 317) water is brought to the top of the wheel and there enters buckets fastened all round the rim. The water remains in these

buckets until the wheel, turned by the extra weight of water on one side, has brought the buckets into such a position that the water is spilled out. The wheel is thus rotated continuously and drives machinery by means of toothed wheel gearing.

In **breast-shot wheels** the water enters the buckets about half-way up, and the action is similar to that in over-shot wheels. In both these types, the attempt is to utilise the potential energy of the water only. In **under-shot wheels** the wheel is furnished with blades, and the water is caused to impinge on these near the bottom of the wheel. The water entering the wheel must have considerable speed, and its kinetic energy is utilised.

Water-wheels are seldom constructed now; they waste a large amount of the available energy and are not suitable for developing large powers.

**Water-turbines.**—The modern system of utilising the energy of elevated water is by the employment of **turbines**. In these machines the water passes through a wheel furnished with blades. The action consists in causing the water to whirl before entering the wheel; in this condition it possesses angular momentum, and the function of the wheel blades is to abstract the angular momentum and to discharge the water with no whirl. A couple will thus act on the wheel (p. 184), and will cause it to rotate, thus performing mechanical work.

In **impulse turbines** arrangements are made so as to convert the whole of the available energy of the water into the kinetic form before it enters the wheel. In **reaction turbines** the energy is partly in the kinetic form and partly in the form of pressure energy.

The action in the **Girard impulse turbine** may be understood by reference to Fig. 318. Water is supplied from A and passes through a ring

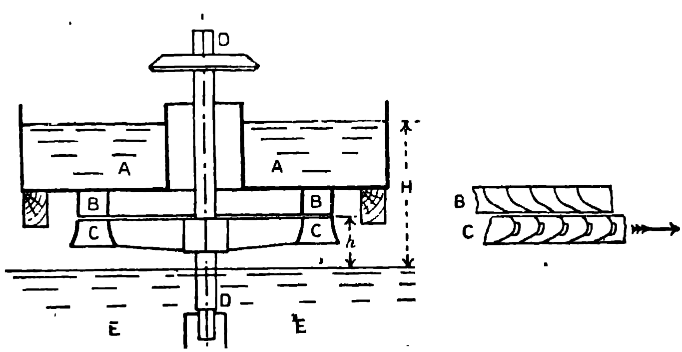


Fig. 318.—Action of a Girard impulse turbine.

of guide passages B, B, having blades so shaped as to cause the water to whirl. Immediately under the guide passages is a horizontal wheel C, which is fixed to a vertical shaft DD. This wheel has a ring of blades

round its rim bent contrary to the blades in the guide passages. If the wheel were prevented from rotating, the action of the wheel blades would be to direct the water backwards. The wheel actually revolves in the direction shown by the arrow, and the effect is to cause the water to be discharged vertically downwards from the wheel; the whirl is thus eliminated. The water leaves the guide blades B, B in a ring of jets under atmospheric pressure; hence the potential energy—represented by $(H - h)$ units per unit mass of water—has been converted into kinetic energy in the jets. The water passes in thin layers over the wheel blades C, C, and the pressure in the wheel passages is kept equal to that of the atmosphere by means of side openings in the rim of the wheel, one at the back of each blade. It will be noted that the wheel is situated above the level of the discharged water in the tail-race E, E; the water is therefore discharged at atmospheric pressure from the wheel into the atmosphere.

In Fig. 319 is shown in outline a **Jonval reaction turbine.** The arrangement is similar to the Girard turbine. Water is supplied from A and

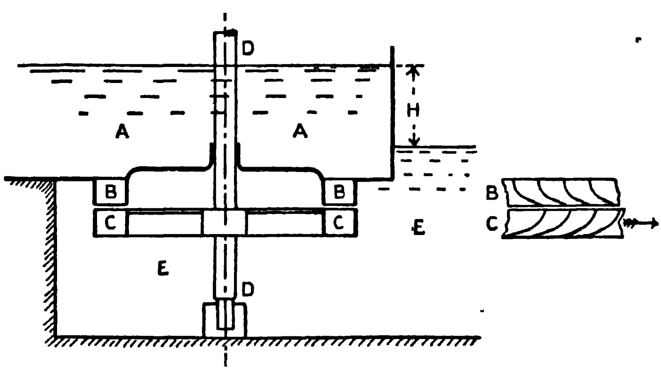


FIG. 319.—Action in a Jonval reaction turbine.

passes through a ring of orifices B, B, having guide blades so as to whirl the water. The wheel C, C has blades so shaped as to eliminate the whirl. The difference between the two types is that in the Jonval turbine the water passing through the wheel fills completely the passages in the wheel, and may therefore have a pressure not equal to that of the atmosphere. In the example illustrated the wheel is below the level of the water in the tail-race E, E, and the pressure in the wheel passages is therefore greater than that of the atmosphere.

The difference in free surface levels of the supply water in A, A and of the discharged water in E, E is H; hence H units of potential energy per unit mass of water are available for conversion into work.

**Pelton wheel.**—To obtain efficient conditions of working in water turbines, the wheel blades must be so formed that the water slides on to them without impact. Impact, or shock, always produces waste of energy (p. 219). Further, the water must be discharged from the wheel

with as small a velocity as possible. Both of these conditions will be readily understood by reference to Fig. 320 showing a **Pelton wheel.** A jet of water is discharged into buckets which are fixed to the rim of a revolving wheel. In the plan the buckets are made double, having a sharpened dividing edge; the jet enters the buckets at this edge and divides, part flowing round one bucket and part flowing round the other. There is thus no shock produced by the entering water, which slides tangentially into the buckets. If the wheel were at rest, the water leaving the buckets would have a velocity in the direction opposite to that of the water in the jet. Owing, however, to the velocity of the bucket, the water leaving the bucket has very little velocity relative to the earth. If the velocities of the jet and the bucket are $v_1$ and V respectively, and if V is equal to $\frac{1}{2}v_1$, then the velocity of the leaving water relative to the earth will be zero, and the whole of the kinetic energy of the water in the jet is available for conversion into work. If $m$ be the mass of water per second delivered by the jet, then

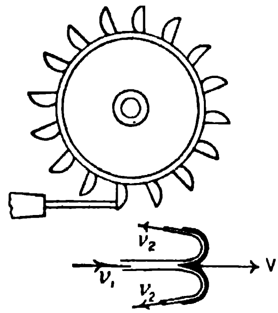


FIG. 320.—Action in a Pelton wheel.

$$\text{Energy supplied per sec.} = mv_1$$

In practice from 70 to 90 per cent. of this appears as useful work done on the wheel.

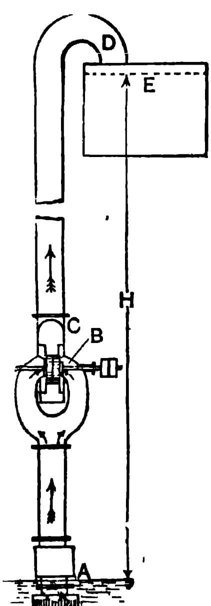


FIG. 321.—Arrangement of a centrifugal pump.

**Centrifugal pumps.**—Water may be raised from a lower to a higher level by means of a **centrifugal pump.** In Fig. 321 the water in A flows up a vertical pipe, and reaches a wheel B where additional kinetic energy is imparted to it. The wheel is driven by some source of power and whirls the water, giving it a higher speed. This speed is reduced gradually in the casing which surrounds the wheel, and hence the kinetic energy added by the action of the wheel is converted into pressure energy in the discharge pipe at C. The resulting pressure is sufficient to overcome the head of water in the pipe CD, hence flow is maintained upwards, and the pressure energy is converted finally into potential energy in the upper tank E. If H is the difference in free surface levels, then $g$H units of useful work have been done per unit mass of water.

**Viscosity.**—It is common experience that when a fluid is disturbed so that the various parts of it have different velocities, and the cause of disturbance is removed, the fluid comes to rest. Thus relative motion of the parts implies some force resisting this motion. According to Newton, there is a shearing stress in a liquid when it is in motion in parallel layers of differing velocity, the stress being proportional to the velocity gradient and having a definite value for each liquid, which value decreases with rise of temperature. Imagine a liquid resting upon a plane solid surface (Fig. 322). The liquid in contact with it is at rest, and if the layers above it are in motion parallel to it, the velocity increases in passing upwards. Suppose that the velocity increases from $v$ to $v+dv$ in passing upwards from B to A through a distance $dx$.

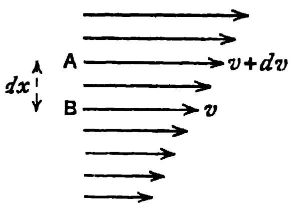


FIG. 322.—Flow of a liquid.

Then the velocity gradient is $\frac{dv}{dx}$, and there is a shearing stress proportional to this gradient for any given liquid. For any layer the shearing stress is a force per unit area producing a drag upon the faster moving layer and an equal pull forward upon the more slowly moving layer, thus tending to bring them to a common velocity. The ratio of this tangential stress to the velocity gradient is called the **coefficient of viscosity**, $\eta$, or more shortly, the **viscosity** of the liquid.

Thus, $$\eta = \frac{f}{\frac{dv}{dx}}, \quad \text{or,} \quad f = \eta \frac{dv}{dx}.$$

The dimensions of $\eta$ are $\frac{mlt^{-2}}{l^2} \Big/ \frac{l}{t \times l}$, or, $ml^{-1}t^{-1}$.

On the C.G.S. system the unit of viscosity is called the **poise**. It is the tangential force over 1 sq. cm. of interface between two layers of fluid when the velocity gradient is 1 cm. per sec. for each centimetre measured at right angles to the flow.

**Flow of liquid in a tube (Poiseuille).**—When the flow of liquid in a tube is stream line (p. 267), the velocity increases from zero in contact with the tube to a maximum at the axis. The first step in determining the flow is to find an expression for the equilibrium of a thin cylindrical layer when the flow in the tube has become steady. Consider the cylindrical layer AB (Fig. 323). At any cylindrical surface there is a stress of $\eta \frac{dv}{dr}$ dynes per sq. cm. due to the velocity gradient and, from

symmetry, this is the same over the whole cylinder, whose area is $2\pi r l$, and the total force at the cylindrical surface is therefore $2\pi r l \eta \frac{dv}{dr} = X$.

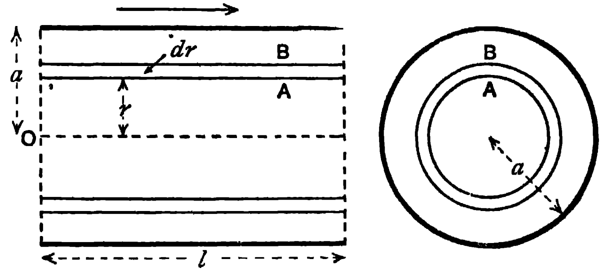


FIG. 323.—Flow of liquid in a tube.

Let this have the value $X_A$ at the inner surface A of the layer and $X_B$ at the outer surface B. $X_B$ causes a check on the motion of the liquid within AB, while $X_A$ pulls it forward, because the liquid nearer the axis is moving the faster. The difference, or, $X_B - X_A$, is balanced by the difference of pressure between the ends of the tube when the flow is steady. If this difference of pressure is $p$ dynes per sq. cm., the resultant force driving the layer forwards is $p \cdot 2\pi r \cdot dr$, the area of cross-section of the layer being $2\pi r \cdot dr$. Note that $p$ is constant for all distances from the axis because there is no flow perpendicular to the axis. Then for steady motion :

$$X_B - X_A = p \cdot 2\pi r \cdot dr.$$

The next step is to find how the quantity X varies across the whole section of the tube. For the small distance $dr$, $X_B - X_A$ may be written $dX$, and the equation of equilibrium is then

$$dX = 2\pi p r \cdot dr,$$

which on integrating becomes $X = \pi p r^2 + C_1$,

where $C_1$ is some constant whose value may be found by noting that at the axis $\frac{dv}{dr} = 0$, since the velocity on one side of the axis is the same as on the other. Thus when $r = 0$, $X = 0$, $\therefore$ $C_1 = 0$, and $X = \pi p r^2$.

Now replacing X by its value $2\pi r l \eta \frac{dv}{dr}$,

$$2\pi r l \eta \frac{dv}{dr} = \pi p r^2,$$

or,

$$dv = \frac{p}{2l\eta} r \cdot dr,$$

and integrating,

$$v = \frac{p}{4l\eta} r^2 + C_2.$$

Since the liquid in contact with the tube does not move, $v=0$ when $r=a$, the radius of the tube.

$$\therefore\ C_2 = -\frac{p}{4l\eta}a^2,$$

and,
$$v=\frac{p}{4l\eta}(r^2-a^2).$$

Having found the velocity of the liquid at every point within the tube, it only remains to find the total volume of liquid flowing through the tube per second. If $dV$ is the volume per second flowing through the layer AB,

$$\begin{aligned} dV &= 2\pi r\,.\,dr\,.\,v \\ &= \frac{\pi p}{2l\eta}(r^3-a^2r)\,dr\,, \\ \therefore\ V &= \frac{\pi p}{2l\eta}\int_0^a (r^3-a^2r)\,dr \\ &= \frac{\pi p}{2l\eta}\left[\frac{r^4}{4}-\frac{a^2r^2}{2}\right]_0^a \\ &= \frac{\pi p a^4}{8l\eta}\,; \\ \therefore\ \eta &= \frac{\pi p a^4}{8lV}\,. \end{aligned}$$

EXPT. 51.—**Measurement of viscosity.** A capillary tube AB (Fig. 324) is placed horizontally with the end A leading from a vessel of water. This is arranged with a vertical tube CD so that a constant head of water $h$ is maintained at the end A of the capillary tube. Since the end B of the tube is open to the atmosphere, the value of $p$ in the above equation for $\eta$ is $hg$, assuming unit density for water. The weight of water collected in the vessel E in a given time is found, so that the volume V passing through the tube per second is known. The radius $a$ is found by placing a thread of mercury in the tube and measuring the length of the thread and its weight. Knowing the density of mercury the volume of the thread is found and, dividing by its length, the area of cross-section $\pi a^2$ is known, from which $a$ is calculated. The water must pass slowly through the tube, so that no appreciable error is introduced on account of the emerging water possessing kinetic energy. If great accuracy is required, the velocity of emergence must be calculated and a corresponding correction made for $h$ (p. 269).

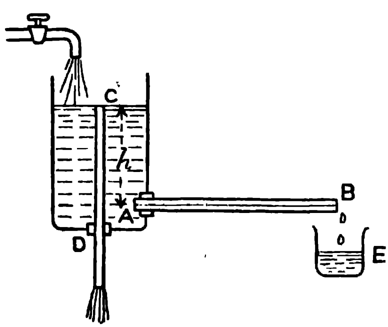

FIG. 324.—Measurement of viscosity.

Also the temperature of the water should be found and the appropriate density employed in place of unity.

## Exercises on Chapter XXI

**1.** Describe what is meant by steady and unsteady motion in fluids. Explain what is meant by a stream line.

**2.** Describe briefly Osborne Reynolds's colour band experiment. What is meant by the critical velocity?

**3.** Water is travelling through a bent pipe (Fig. 325), and it is found that the fluid pressure on the wall at A is greater than that at B. Explain this clearly.

**4.** Water is flowing steadily along a pipe. Calculate the total energy possessed by one pound of the water at a point where the pressure is 30 lb. wt. per square inch, the velocity is 4 feet per second, and the height is 16 feet above ground level.

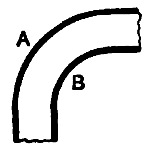


FIG. 325.

**5.** State Bernoulli's theorem. When a liquid flows along a horizontal pipe having a gradual constriction, the pressure at the constriction is less than that at the larger parts of the pipe. Explain this, and describe briefly an experiment for demonstrating it.

**6.** Water flows up a vertical pipe from ground level to a point 40 feet above the ground. The speed is constant and is 6 feet per second. The top of the pipe is open to the atmosphere. Find the potential, pressure and kinetic energies of one pound of water at points (*a*) at the top of the pipe, (*b*) at 6 feet above ground level.

**7.** Water is flowing steadily along a horizontal pipe of varying section. At a place where the pressure is 20 lb. wt. per square inch the speed is 4 feet per second. At another place the speed is 40 feet per second. What is the pressure at this place?

**8.** If a liquid flows steadily through a pipe of varying circular cross-section, show that the speed is inversely proportional to the square of the diameter of the pipe provided that the liquid fills the pipe completely.

**9.** A horizontal pipe of circular section is 4 inches in internal diameter at a section A and contracts to 1 inch diameter at another section B. Water flows steadily along the pipe, filling it completely, and has a speed of 4 feet per second at A. If the pressure at A is 40 lb. wt. per square inch, find the pressure at B, neglecting friction.

**10.** Give a brief general account of the changes in energy which occur when one pound of water passes from the free surface level in a tank, through the tank and is finally discharged through an orifice in the side of the tank.

**11.** A tank containing water has an orifice in one vertical side. If the centre of the orifice is 9 feet below the free surface level in the tank, find the velocity of discharge, assuming that there is no wasted energy. The actual velocity is 97 per cent. of the value calculated above; find the actual velocity.

**12.** A circular jet of water is 0·4 inch in diameter, and has a speed of 30 feet per second. Calculate the quantity in cubic feet which passes any given section in one second.

**13.** A tank contains water of which the free surface level is maintained constantly at 4 feet above the centre of a sharp-edged orifice in the side of the tank. The orifice is 1 inch in diameter. Take the usual values for the various coefficients (p. 272) and calculate (*a*) the actual velocity of the jet at the section where contraction is complete, (*b*) the diameter of the jet there, (*c*) the volume discharged per second in cubic feet.

**14.** A stream of water supplies an over-shot water-wheel which is 20 feet in diameter. The stream is 4 feet wide and 6 inches deep, and flows at 6 feet per second. Calculate the weight of water supplied per minute. If 65 per cent. of the potential energy of the water alone is converted into useful work, find the horse-power developed by the wheel.

**15.** Distinguish between impulse and reaction water turbines. Give clear sketches and a very brief description of the action in a turbine of each of these types.

**16.** A Pelton wheel is supplied by a jet of water 4 inches in diameter and having a speed of 120 feet per second. How much energy is supplied per second? If the efficiency is 80 per cent., what horse-power can the wheel develop?

**17.** Describe briefly, by reference to a sketch, the action of a centrifugal pump.

**18.** Describe the action of a siphon. Give any practical application you may have observed of the use of a siphon.

**19.** A water pipe of circular section whose internal diameter is 12 inches bends through an angle of 10 degrees remaining in one horizontal plane. Water flows through the pipe at a mean speed of $V$ feet per second. Prove that the total horizontal thrust on the bend is proportional to $V^2$. Find its magnitude in lb. wt. when $V = 8$. (1 c. ft. of water weighs 62·5 lb. Take $\pi = 22/7$ ; $g = 32$.) J.M.B., H.S.C.

**20.** Show that when a particle executes simple harmonic motion the ratio of the acceleration to the displacement from the mean position at any time is equal to $4\pi^2/T^2$, where $T$ is the period of vibration.

Hence find an expression for the time of vibration of a Nicholson's hydrometer of mass $M$, when the instrument floating in a liquid such as water is given a small downward displacement and is then released, supposing that the cross-section of the neck of the instrument is $s$ and the density of the liquid is $d$. C.W.B., H.C.

**21.** What is meant by *viscosity*?

Explain the forces which are called into play when two parallel layers of liquid are set in motion relative to each other in a direction parallel to the planes.

Show how similar forces act in the case of a liquid flowing through a narrow-bored tube, and indicate the factors which determine the quantity of liquid which passes per second through such a tube placed horizontally. L.U.

# CHAPTER XXII

## SURFACE TENSION. DIFFUSION. OSMOSIS

**Surface tension.**—It is a matter of common observation that a drop of liquid, *e.g.* water, can cling to the lower side of a horizontal glass plate. This fact illustrates two properties: the liquid can adhere to the glass by reason of molecular attraction between the substances, and the liquid behaves as though it were enclosed in an elastic bag, having a constant tendency to contract, and forming a boundary between the liquid and the atmosphere which surrounds the drop. Water or other liquid in an open vessel has a horizontal free surface and this surface shows properties similar to those of a stretched elastic film. Thus a clean, dry needle may float on the surface of water, and is supported by the action of the surface film which bends under the weight of the needle.

The portions of the free surface of any liquid in an open vessel lying on opposite sides of any straight line, drawn in the surface, resist separation, showing the existence of tension in the surface. The **surface tension** is measured by the force in dynes exerted across a portion of the line one centimetre in length.

EXPT. 52.—**Surface tension of water.** Make a rectangular frame of platinum wire (Fig. 326) about 3 cm. long and 1·5 cm. high. Clean the frame by heating it in a Bunsen flame and hang it from one arm of a balance, and let the top be about 3 mm. above the surface of water in a beaker. Add weights to the balance so as to restore equilibrium. Depress the arm of the balance from which the frame hangs so as to immerse the frame. On allowing the frame to rise again, it will be found that it has taken up a film of water, and that more weights are required in order to restore equilibrium. By taking the difference in weights, obtain the total pull of the film, P grams weight, say. The film of water has two surfaces, front and back; hence the surface tension T is calculated from

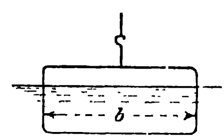

FIG. 326.—Measurement of the surface tension of water.

$$T = \frac{Pg}{2b} \text{ dynes per centimetre,}$$

where $b$ is the breadth of the frame in centimetres.

The surface tension of water is 75·8 dynes per centimetre at 0° C., and decreases by 0·152 dyne per cm. for each degree rise of temperature. Take the temperature of the water in the beaker at the time of performing

the experiment; estimate the surface tension, and compare the result with that obtained in the experiment.

**Work done in producing surface.**—The production of a liquid surface requires work. Let a film be formed in a wire network ABCD (Fig. 327), of which the side AD can slide over BA and CD. The force F required to act on AB in order to maintain it at rest in opposition to the surface tension T in the film is $2Tl$, remembering that the film has two surfaces. Now let F move the wire slowly through distance $x$ into the position GH. The work done is $Fx = 2Tlx$ ergs. The area of surface is increased in this movement by the amount $2lx$. Hence the work done to produce unit area of surface is T ergs.

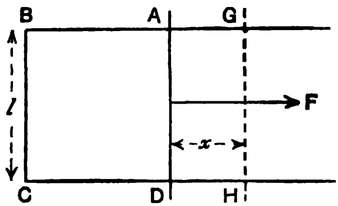

FIG. 327.—Extension of film.

The surface tension must not be called the surface energy of the film, because this involves also the heat that must be supplied to maintain the film at constant temperature while it is being stretched. It is merely the mechanical energy per unit area required to increase the surface.

**Angle of contact.**—If three surfaces of separation meet at a line, there are three surface tensions to consider. In Fig. 328 suppose B and C to be two liquids in contact with each other and with air along a line at right angles to the plane of the diagram and represented by P. If $T_1$ is the surface tension of B in contact with air, $T_2$ that for C and air, and $T_3$ that for B in contact with C, then $T_1$, $T_2$ and $T_3$ are three forces acting upon 1 cm. of the line of mutual contact. If the three forces produce equilibrium they can be represented by a triangle of forces (p. 69). But if one of the forces is greater than the sum of the other two, it is impossible to construct such a triangle, and one of the liquids spreads out indefinitely upon the surface of the other as oil spreads upon water.

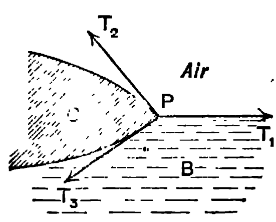

FIG. 328.—Two liquids in contact with air.

When a solid replaces the liquid B (Fig. 328) the surface tensions still exist. Thus the surface tension of molten glass is evident to anyone who does glass-blowing, and there is no reason to suppose that the surface tension disappears when the glass solidifies. In fact surface tension increases with fall of temperature, although the rigidity of the solid glass prevents many of the phenomena observed in liquids from being seen.

Imagine a liquid to rest on a solid, as in Fig. 329. Then if $T_2 > T_1 + T_3$, there is no possible position of equilibrium and the liquid spreads over the solid, as water spreads over clean glass. But if the glass surface is greasy, $T_2$ is very much reduced, and it may then be possible to construct a triangle of forces of $T_1$, $T_2$ and $T_3$. The water will then stand in drops upon the contaminated surface. In such a case $T_2 = T_1 + T_3 \cos\theta$, where $\theta$ is the internal angle which the liquid makes with the solid at contact. Generally, $T_1 > T_2$, in which case $\cos\theta$ is negative and $\theta$ lies between 90° and 180°. For mercury and clean glass, $\theta$ is about 140°. For water and many other liquids on clean glass, $\theta = 0°$, and the liquid is said to wet the glass.

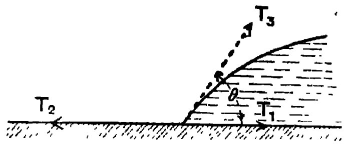

FIG. 329.—Angle of contact.

A method of measuring the angle of contact of mercury on glass is illustrated in Fig. 330. The glass plate AE is slanted until, at the contact with the mercury at C, the surface of the mercury is without curvature. This may be seen by reflected light. Then on measuring AB and BC, the angle at C may be found and the internal angle of contact calculated.

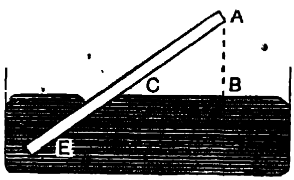

FIG 330.—Angle of contact.

**Pressure due to a curved surface.**—When the surface of a liquid is plane, the surface tension merely tends to prevent stretching of the surface. But if the surface is curved, there is a force due to tension pulling the surface inwards. This must be balanced by a pressure exerted outwards when equilibrium is maintained. If O (Fig. 331) is a small element of the surface of cylindrical shape, of radius $r$, the force along each edge of unit length is T. The two forces T and T have a resultant $OP = T \, . \, d\theta$, pressing the element of surface inwards. This is balanced by the force on the element due to the pressure $p$ on the inner side of the surface whose area is $1 \times r \, . \, d\theta$.

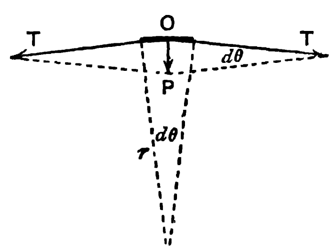

FIG. 331.—Pressure due to curvature.

$$\therefore \; pr \, . \, d\theta = T \, . \, d\theta,$$

or,
$$p = \frac{T}{r}.$$

If the surface is part of a sphere instead of a cylinder, the curvatures

in two directions each contribute the amount $\frac{T}{r}$ to the pressure, so that the pressure inside a spherical surface is $\frac{2T}{r}$ for equilibrium.

Remembering that a spherical soap bubble has two surfaces, an inner and an outer, and each contributes the same amount of pressure, the pressure of the air inside such a bubble must exceed the pressure outside by $\frac{4T}{r}$.

This result may also be obtained by the principle of virtual work (p. 147). If $p$ is the excess pressure of air inside the bubble, the total work done by the pressure when the bubble increases in radius by the small amount $dr$ is (area of bubble) $\times p \, . \, dr = 4\pi r^2 p \, . \, dr$. Now the work done in increasing the area of liquid surface is $\{4\pi(r+dr)^2 - 4\pi r^2\}T$ for the inside surface and the same for the outside surface, or,

$$2\{4\pi(r^2 + 2r \, . \, dr + (dr)^2) - 4\pi r^2\}T$$

in all. This is $16\pi r T \, . \, dr$, neglecting $(dr)^2$ as infinitesimal. Thus by the principle of virtual work,

$$16\pi r T \, . \, dr = 4\pi r^2 p \, . \, dr,$$

or,

$$p = \frac{4T}{r}.$$

**Capillary elevation.**—If a glass tube of fine bore, open at both ends, be dipped vertically into water it will be observed that some of the water rises in the tube to a level higher than the free surface outside the tube, and that the surface of the water in the tube which is exposed to the pressure of the atmosphere is shaped like a cup (Fig. 332). This cup is termed the meniscus. The elevation of the water inside the tube appears to controvert the laws of fluid pressure and is attributable to surface tension. Water wets glass and tends to spread over its surface; the tendency of the surface skin to contract is resisted by the weight of the water in the glass tube.

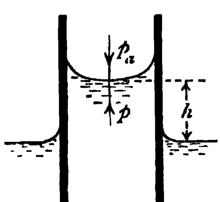

FIG. 332.—Capillary elevation.

The shape of the surface may be explained by considering that the elastic surface skin is subjected on the upper side to the pressure of the atmosphere $p_a$, and, on the lower side, to a pressure $p$ which is less than $p_a$ by an amount corresponding to the difference in head $h$. The superior pressure $p_a$ therefore causes the skin to bulge downwards. If $d$ be the density of the liquid, then the difference in the pressures on the opposite sides of the skin is $hdg$ dynes per square centimetre. If

the tube has a radius $r$ centimetres, then the area over which the pressure is distributed is $\pi r^2$, and the resultant vertical force acting on the surface skin is given by

$$P = hdg\pi r^2. \qquad (1)$$

This force is balanced by the surface tension T distributed round the inner boundary of the tube, of length $2\pi r$, and since the liquid wets the tube, the surface tensions at this boundary are upward vertical forces. Hence

$$T \times 2\pi r = P = hdg\pi r^2,$$

or,

$$T = hdgr/2. \qquad (2)$$

If the tube is of small diameter, the surface of the meniscus is very nearly hemispherical. The volume of water above a horizontal plane which touches the meniscus at its lowest point will be the difference between the volume of a cylinder of radius $r$ and height $r$, viz. $\pi r^3$, and the volume of a hemisphere of radius $r$, viz. $\frac{2}{3}\pi r^3$, and is therefore $\frac{1}{3}\pi r^3$. The error in measuring $h$ to the bottom of the meniscus may therefore be corrected in tubes of small bore by adding $\frac{1}{3}r$ to the height $h$.

If a similar experiment be tried with mercury, it will be found that the surface of the mercury inside the tube is depressed below the level of the free surface outside the tube (Fig. 333). Mercury does not wet glass, and in this case the skin is bulged upwards by reason of the pressure $p$ on the lower side of the skin being greater than the atmospheric pressure $p_a$ on the upper side. Mercury has a definite angle of contact $\alpha$ with glass (about 140° measured internally or 40° measured externally), and hence it is necessary in this case to take the vertical components of T round the boundary. Thus

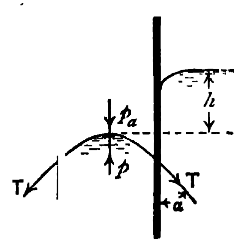

FIG. 333.—Capillary depression of mercury.

$$T \cos\alpha \times 2\pi r = P = hdg\pi r^2,$$

$$T = hdgr/2 \cos\alpha. \qquad (3)$$

The surface tension of mercury is 547 dynes per cm. at 17·5° C., and diminishes by 0·379 dyne per cm. for each degree C. rise in temperature. The external angle of contact varies considerably, depending on the freshness of the surfaces; it is 41° 5′ in a freshly formed drop on glass, and may increase to 52° 40′ for surfaces which are not fresh. Fouling of the glass in mercurial barometers accounts for the fact that the shape of the meniscus in a rising barometer differs from that when the barometer is falling.

EXPT. 53.—**Measurement of the surface tension of water by the capillary tube method.** Clean the tubes supplied by drawing through them strong sulphuric acid and then washing with distilled water. Point one end of a piece of wire, bend it twice at right angles and secure it to one of the

tubes by means of rubber bands (Fig. 334). Fix the tube vertically and let the lower end dip into a beaker of water; the beaker should rest on a support so that it may be removed easily without disturbing the tube. Adjust the height of the tube until the point of the wire lies exactly in the surface of the water; the point should not be too close to the tube or the side of the beaker. Attach a piece of rubber tubing to the top of the glass tube, and draw water up the tube so as to wet the interior.

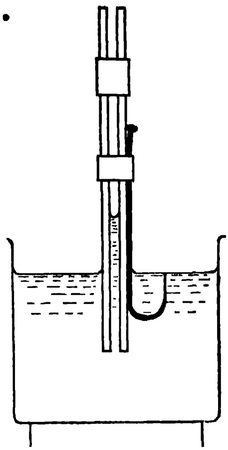

FIG. 334.—Surface tension of water by capillary tube method.

Focus a vernier microscope on the liquid in the tube, and take the reading corresponding to the bottom of the meniscus. Remove the beaker, and by means of the microscope obtain the reading corresponding to the point of the wire. The difference in these readings will give the elevation of the water in the tube above the free surface level in the beaker.

Repeat the experiment with several tubes of different diameter; in each case measure the diameter of the tube (Expt. 7, p. 14, or Expt. 51, p. 278), and note the temperature of the water in the beaker.

Calculate the value of the surface tension in each experiment, using equation (2), p. 285. Apply the correction for the shape of the meniscus.

If the relation between surface tension and temperature is required, the capillary may be in the form of a U tube (Fig. 335). The fine bore tube AB is fused on to the tube CD of wider bore, and the difference of level of meniscus $h$ in the two is found. If $r_1$ is the radius of bore of AB and $r_2$ that of CD, then since both tubes are open to the atmosphere the pressure above the liquid columns is the same. The pressure in the liquid in AB just under the surface is $\frac{2T}{r_1}$ below the atmospheric pressure (p. 284), and that in CD is $\frac{2T}{r_2}$, so that the difference of pressure $\frac{2T}{r_1} - \frac{2T}{r_2}$ is equal to that of the column of liquid $h$,

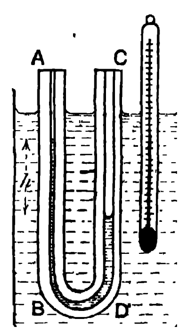


FIG. 335.—Surface tension and temperature.

$$\therefore \; 2T\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = h\rho g,$$

where $\rho$ is the density of the liquid.

The method may also be used for mercury if the angle of contact $\theta$ has previously been found (p. 283), in which case the lower level is in the smaller tube.

$$\frac{2T\cos\theta}{r_1} - \frac{2T\cos\theta}{r_2} = h\rho g,$$

or,
$$2T\cos\theta\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = h\rho g.$$

**Jaeger's method.**—If the open end of a tube is situated under the surface of a liquid, pressure is required in order to force the air in the tube down to the level of the opening. If the air is then to emerge from the tube into the liquid in the form of bubbles, additional pressure is required to overcome the surface tension of the liquid. The former is $h_2\rho_2 g$, where $\rho_2$ is the density of the liquid (Fig. 336). The additional pressure is $\frac{2T}{r}$ (p. 284), where $r$ is the radius of the opening of the tube at A. Before the bubble has reached hemispherical form, the pressure must rise in order to increase the radius of the bubble. But after this stage is reached the bubble is unstable, and blows out quickly and becomes detached. For the limiting state $h_2\rho_2 g + \frac{2T}{r} = h_1\rho_1 g$, where $h_1\rho_1 g$ is the total pressure as measured by the gauge D, $\rho_1$ being the density of the liquid in D. The height $h$, for which the bubbles become detached, is observed, the reservoir B supplying the air under pressure, and the tap C regulating its passage to A.

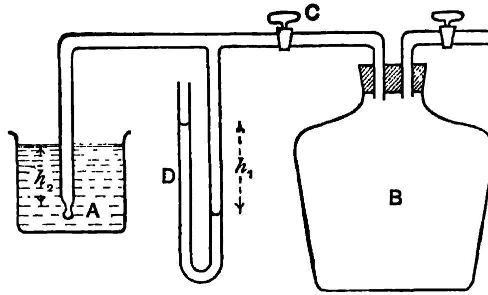


FIG. 336.—Jaeger's method.

Then, $\quad T = \frac{rg}{2}(h_1\rho_1 - h_2\rho_2).$

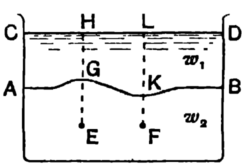


FIG. 337.—Surface of separation in non-mixing liquids.

**Liquids which do not mix.**—In Fig. 337 is shown a vessel containing two liquids which do not mix. Suppose AGKB to be the surface of separation of the liquids, and consider two points E and F in the same horizontal plane. Let $w_1$ and $w_2$ be the weights per unit volume of the upper and lower liquids respectively. The pressures at E and F must be equal; hence

$$(w_1 \times HG) + (w_2 \times GE) = (w_1 \times LK) + (w_2 \times KF);$$

$$\therefore\ w_1(HG - LK) = w_2(KF - GE). \quad\ldots\ldots\ldots\ldots(1)$$

Also, $$HG + GE = LK + KF;$$

$$\therefore\ HG - LK = KF - GE. \qquad (2)$$

For (1) and (2) to be true simultaneously, either $w_1$ and $w_2$ must be equal, in which case both liquids have the same specific gravity, or if $w_1$ and $w_2$ be unequal, then the result of (2) must be zero, *i.e.*

$$HG = LK, \quad \text{and} \quad KF = GE.$$

Hence the surface of separation must be parallel to the free surface CD, and must therefore be a horizontal plane.

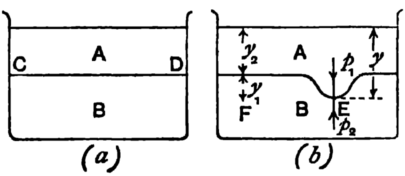


FIG. 338.—The heavier liquid must occupy the lower part.

In Fig. 338 (*a*) the heavier liquid A is supposed to occupy the upper part of the vessel. That the equilibrium is unstable may be shown as follows: Let the surface of separation be disturbed as shown in Fig. 338 (*b*) and consider a small area on this surface at E. The pressure $p_1$ on the upper side is given by

$$p_1 = w_A y. \qquad (1)$$

Take another point F in the same horizontal plane as E. The pressures $p_2$ at E and F are equal, and are given by

$$p_2 = p_F = w_B y_1 + w_A y_2. \qquad (2)$$

Also, $$y = y_1 + y_2;$$

$$\therefore\ p_1 = w_A y_1 + w_A y_2 \text{ (from (1))}. \qquad (3)$$

Comparing (2) and (3), and remembering that $w_A$ is greater than $w_B$, we see that $p_1$ is greater than $p_2$. Hence on the small area at E there is a resultant downward pressure $(p_1 - p_2)$. Therefore the disturbance at E will continue downwards, and the heavier liquid will occupy ultimately the lower part of the vessel. The state of equilibrium shown in Fig. 338 (*a*) is therefore unstable.

The same principle also applies to gases. Carbon dioxide has a density greater than that of air, and therefore tends to occupy the lower part of an enclosed space. This fact has been illustrated by the death of small animals in vats containing some carbon dioxide, while men have been able to breathe the superstratum of air. Stratification of this kind is not permanent; **diffusion** takes place more or less quickly, and produces an atmosphere in which both gases are distributed uniformly.

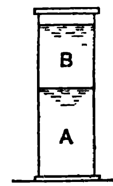


FIG. 339.—Diffusion of liquids.

**Diffusion of liquids.**—In Fig. 339 is shown a jar containing two liquids A and B, A having a greater density than B. If the liquids are incapable of mixing, no alteration will take place if the jar is left undisturbed; but if the liquids possess the capability

of mixing in any proportion, it will be found that a process of self-mixing is going on, A travelling upwards in spite of its greater density, and B travelling downwards. Finally the mixture becomes uniform throughout the jar. This process is called **diffusion.**

Diffusion in liquids takes a long time to complete. A demonstration jar may be prepared by introducing a strong solution of copper sulphate A (Fig. 339), the quantity being rather less than half the capacity of the jar. An equal quantity of distilled water B is then poured in carefully so as not to disturb the copper sulphate. The jar should be covered and placed where it will not be disturbed. Periodic inspections will show that the blue colour of the copper sulphate is extending upwards, and that the tint in the lower part of the vessel is becoming fainter. At one stage the colour gradation extends throughout the whole depth of liquid. Finally, uniformity of tint is attained, showing that diffusion is complete.

Observations in experiments of this kind show that the time required to complete the diffusion process is proportional to the square of the total depth of liquid. Solutions of different substances, having the same degree of concentration and other conditions similar, have been found to possess different rates of diffusion ; for example, hydrochloric acid diffuses more rapidly than potassium bromide. Solutions of the same substance, having different degrees of concentration, have been found to possess rates of diffusion proportional to the strength of the solution. Increase in temperature increases considerably the rate of diffusion.

Diffusion can be completed in a few seconds in a jar, such as is shown in Fig. 339, by using a piece of wire having a loop bent at right angles at one end and stirring the liquids vertically. The effect of such stirring is two-fold ; layers of strong solution are brought into juxtaposition with layers of water, and therefore the rate of diffusion is greatly increased ; further, the concentrated layers of solution have now a shorter distance to travel in completing the diffusion process.

The uniformity of distribution of the various substances dissolved in sea water is owing to diffusion. Otherwise the ocean would consist of stratifications of salt solutions of different densities, the heaviest being at the bottom.

**Diffusion of gases.**—Gases possess the property of diffusion, and the process is completed much more rapidly than is the case with liquids.

EXPT. 54.—**Diffusion of gases.** Referring to Fig. 340, A is a flask charged with coal-gas and B is another flask having a capacity about eight times that of A. The flasks are fitted with rubber stoppers, and are connected by means of a glass tube about 18 inches long and ¼ inch bore. Leave the

arrangement undisturbed in a vertical position, as shown in Fig. 340, for two or three hours. *It will be found that diffusion has taken place, the heavier air in B travelling upwards and the lighter gas in A downwards.* That the gases have mixed may be proved from the fact that a gaseous mixture of air and coal-gas having the stated proportions (about eight to one) is explosive. Wrap a piece of cloth round each flask; quickly remove the stoppers, and test each flask by applying a lighted taper.

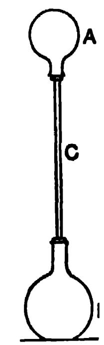

FIG. 340.—Diffusion of gases.

Diffusion in non-uniform mixtures of gases takes place by the flow of each gas from places where its density is higher towards places where its density is lower. Ultimately uniformity of density of each gas throughout the whole space is attained. The rate of diffusion of two given gases depends on the kind of gases; it is inversely proportional to the pressure of the mixed gases, and roughly is proportional to the square of the absolute temperature. The rate of diffusion also depends on the densities of adjacent layers of the two gases; hence mechanical mixing of the gases hastens the process of diffusion, as is the case also in liquids.

The property of diffusion in gases is of great importance in the prevention of accumulations of noxious gases in towns and confined spaces. Carbon dioxide does not support life, and a comparatively small percentage of this gas in the atmosphere is dangerous. The exhalations of animals consist largely of carbon dioxide, which is also given off in large volumes in many industrial processes. The gas diffuses rapidly into the atmosphere, the process being assisted by the stirring produced by air currents, and thus a mixture is attained which is not dangerous. Some idea of the rate of diffusion of carbon dioxide and air may be obtained from the observed fact that in a vertical tube about 60 cm. long, and having the lower tenth of its length charged with carbon dioxide, the upper nine-tenths containing air, diffusion is completed in about two hours. The time taken is proportional to the square of the length of the tube.

**Osmosis.**—The term osmosis is given to the ability which some liquids have to pass through certain membranes. For example, water is able to pass through the membrane of a pig's bladder, while alcohol is unable to do so. Hence, if a pig's bladder be filled with alcohol, closed, and placed under water, it will swell and may burst. If the bladder be filled with water and placed under alcohol, shrinkage occurs. Dried currants placed under water swell and become spherical owing to the passage of water through their skins.

**Expt. 55.—Osmosis.** Arrange apparatus as shown in Fig. 341. A is a glass vessel to which a capillary tube B is attached; the upper end of B is open. The lower end of A is closed by a piece of parchment paper (paper treated with sulphuric acid). Fill A with a solution of sugar so that the level of the liquid is a short distance up the tube, and immerse the vessel in distilled water C, arranging that the liquid levels inside and outside the tube coincide at first. It will be found that the surface level inside B moves upwards with visible velocity, showing that osmotic flow of the water is taking place through the diaphragm into A.

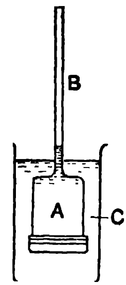


Fig. 341.—Apparatus for demonstrating osmosis.

Graham divided substances into two classes, **crystalloids** and **colloids**. Crystalloids include such substances as glucose, cane sugar, etc.; when dissolved in water, crystalloids can diffuse through a parchment, or animal membrane. Colloids include such substances as gum, starch and albumen; these either do not diffuse at all, or at a very low rate.

These properties led Graham to devise a method of separating crystalloids and colloids from a mixed solution. The method is called **dialysis**. In Fig. 342, A is a tube having its lower end closed by a diaphragm of colloidal substance such as parchment paper or bladder. The mixed solution of crystalloids and colloids is poured into A, and the tube is partially immersed in a vessel of water B. The crystalloids diffuse through the membrane into the water and the colloids remain in A. If the water be changed at intervals, and sufficient time allowed, it is possible to effect nearly complete separation of the colloids from the crystalloids.

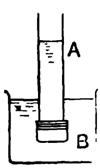


Fig. 342.—Graham's method of dialysis.

The separation produced by dialysis in this way is probably due to the sizes of the constituent particles of crystalloids and of colloids. The view now held is that a colloid particle is an aggregate of molecules too small to be visible in a solution to the unaided eye and yet large enough to affect light and be seen by means of the ultra-microscope. Colloidal solutions may, therefore, be defined as uniform distributions of solids in fluids, which are transparent to ordinary light, and not separable into their constituents by the action of gravity or by filtration. Ruby glass owes its colour to the presence of gold particles in a colloidal state. In the manufacture of the glass, gold chloride is added when the glass is in a molten state. If the glass be cooled quickly, it is colourless, but if it is afterwards heated up to the point of softening it becomes suddenly ruby red. In the coloured glass, the ultra-microscope reveals the presence of colloidal gold particles, but in the colourless glass none can be seen. Colloidal gold can be obtained red, purple, blue or green in solutions containing the same amount of metal, the difference of colour

being due to the difference in the size of the particles, which may vary from 5 $\mu\mu$ to 20$\mu\mu$ ($1\mu\mu = 10^{-7}$ cm.). In recent years much attention has been given to the subject of colloids both in their scientific and their industrial aspects, and Graham's original conception of them has been extended greatly.

**Osmotic pressure.**—In Expt. 55 if the contents of the outer vessel be examined, it will be found that some of the dissolved substance has passed through the membrane. Flow has thus taken place in both directions through the membrane. Parchment paper and bladder permit both crystalloids and water to pass, but there are certain membranes known which will permit water to pass and stop certain salt solutions. For experimental work the most convenient material is the gelatinous precipitate of copper ferrocyanide. This material is very weak, and Pfeffer contrived a method of precipitating it in the interior of the walls of a porous pot, thus producing a continuous film of sufficient strength for practical work.

In Fig. 343, A is a Pfeffer pot with its internal film of copper ferrocyanide B. Into the top of the pot is cemented a glass tube C, which is considerably longer than is shown in Fig. 343. The pot is filled with a dilute solution of salt D, and is then immersed in a vessel E containing distilled water. Inward flow of the water takes place through the pot and its internal film, and the increased bulk of liquid in the pot causes the level to rise in C. The process goes on until a definite pressure is attained in the pot, as indicated by a steady difference in levels in C and E. Inward flow has then ceased.

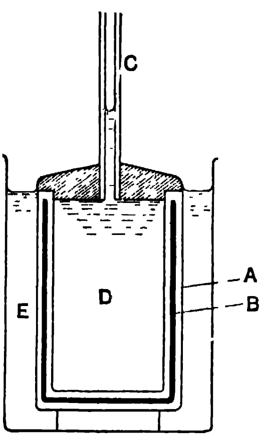

FIG. 343.—Pfeffer pot.

It is evident that had an artificial pressure equal to this final pressure been applied to the contents of the pot, no flow would have taken place. This pressure, which depends on the kind of solution and its strength, is called the osmotic pressure of the solution.

Pfeffer found that the osmotic pressure of dilute sugar solutions is proportional to the mass of sugar per unit volume of the solution. It has also been shown from Pfeffer's results that the osmotic pressure of a given sugar solution is proportional to the absolute temperature (Chap. XXXI).

Let $v$ = volume in c.c. in which a given mass (in grams) of sugar is dissolved.

$p$ = the osmotic pressure in cm. of mercury.

T = absolute temperature = temp. C. + 273.

Then, $$pv = \text{RT} \text{ (compare Chap. XXXI),}$$

where R is a constant which has been shown to have the same value as the sugar would have if it existed as a gas. Thus we may say that the osmotic pressure of a dilute sugar solution is equal to the pressure which the sugar would exert if it were in the gaseous state and occupied a space otherwise empty and equal to the volume of the solution.

EXAMPLE.—A 2 per cent. sugar solution is at a temperature of 37° C. Find the osmotic pressure, assuming hydrogen to have a density of 0·00009 gm. per c.c. at 0° C. and 76 cm. of mercury; molecular weight of hydrogen = 2; of sugar = 342.

$$\text{Density of sugar at } 0^\circ \text{ C. and 76 cm. of mercury} = 0{\cdot}00009 \times \frac{342}{2}$$
$$= 0{\cdot}01548 \text{ gm. per c.c.}$$

$$\text{Volume occupied by 1 gm. of sugar at } 0^\circ \text{ C. and 76 cm.} = \frac{1}{0{\cdot}01548}$$
$$= 64{\cdot}6 \text{ c.c.}$$

$$pv = \mathrm{RT}$$

$$\therefore\ \mathrm{R} = \frac{76 \times 64{\cdot}6}{273} = 17{\cdot}98.$$

$$\text{Osmotic pressure} \times \frac{100}{2} = 17{\cdot}98(273 + 37)$$

$$\therefore\ \text{Osmotic pressure} = \frac{17{\cdot}98 \times 310}{50} = 111{\cdot}5 \text{ cm. of mercury.}$$

**Passage of gases through porous diaphragms.**—The mode by which a gas passes through a porous obstruction depends on the size of the orifices and the thickness of the obstruction. Thus, if the obstruction is thin and the orifice is relatively large, the flow of the gas resembles the flow of a liquid through an orifice in a thin plate (p. 270) and follows the same laws. If the obstruction is thick and the passages still fairly large, the flow of the gas resembles that of a liquid through a capillary tube. If the pores are very fine, such as in plates of plaster of Paris or compressed graphite, the phenomena of flow are quite different from the other two cases. The passages in such a plate are of cross-sectional dimensions comparable with the size of the gaseous molecules, and the flow through any one pore may be regarded as a stream of single molecules following one another in succession.

The laws of flow in such cases were discovered by Graham, who found that the volume, measured at standard pressure, of a given gas passing through a porous plate was directly proportional to the difference in pressure on the two sides of the plate, and inversely proportional to the square root of the molecular weight of the gas. The molecular weights of hydrogen and oxygen are in the proportion of 1 to 16; hence, under like conditions of pressure on the two sides of a porous plate, the rates of flow of hydrogen and oxygen will be in the

proportion of 4 to 1. It therefore follows that, if there be a mixture of stated proportions of hydrogen and oxygen on one side of a porous plate, the mixture after passing through the plate will be found to contain a greater proportion of hydrogen.

EXPT. 56.—**Diffusion of a gas through a porous plug.** In Fig. 344, A is a glass tube having an enlargement near its upper end. Above the enlargement there is a thin plate B of plaster of Paris, and above this again a cork is inserted temporarily. The tube is then filled with hydrogen, and the lower end is inserted in a vessel of water D. On withdrawing the cork C, diffusion of the hydrogen outwards and of the air inwards takes place through the diaphragm. The rate of flow of the hydrogen through the porous plate is much greater than that of the air, on account of its smaller molecular weight; hence it will be observed that the level of the water rises rapidly in the tube.

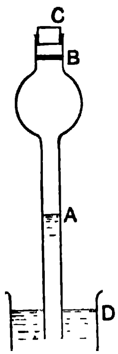

FIG. 344.—Diffusion of gas through a porous plug.

Repeat the experiment, using coal-gas in place of the hydrogen. This gas is a mixture of gases, several of which have molecular weights more nearly approaching those of the mixture of oxygen and nitrogen of which the atmosphere is composed. On the whole, however, the molecular weight of the coal-gas is less than that of the atmosphere, and the flow outwards is therefore greater than that inwards. Hence the level of the water rises, but the velocity is less than when hydrogen is used.

## Exercises on Chapter XXII

**1.** Explain what is meant by the surface tension of a liquid. Give some instances which illustrate the existence of surface tension.

**2.** In an experiment for determining the surface tension of water, performed as directed on p. 281, the breadth of the platinum frame was 2·81 cm. The force required to balance the pull of the film of water was found to be 0·422 gram weight. The temperature of the water was 15° C. Find the surface tension of water at this temperature.

**3.** A capillary tube having an internal diameter of 0·5 mm. dips vertically into a vessel of water. At what height will the water in the tube stand above the surface level of the water in the vessel? Take the surface tension of water to be 73 dynes per cm.

**4.** Give a brief explanation of the shape of the meniscus in tubes containing (*a*) water, (*b*) mercury.

**5.** The limbs of a U tube are vertical, and have internal diameters of 5 and 1 mm. respectively. If the tube contains water, what will be the difference in the surface levels in the limbs? Take the surface tension of water to be 72 dynes per cm.

**6.** A glass tube, 5 mm. in internal diameter, is pushed vertically into mercury. Take the surface tension of mercury to be 545 dynes per cm. and the angle of contact to be 50°. Calculate the difference in level of the mercury in the tube and that outside the tube.

**7.** A ring of glass is cut from a tube 7·4 cm. internal and 7·8 external diameter. This ring, with its lower edge horizontal, is suspended from the arm of a balance so that the lower edge is just immersed in a vessel of water. It is found that an additional weight of 3·62 grams must be placed on the other scale-pan to compensate for the pull of surface tension on the ring. Calculate in dynes per cm. the value of the surface tension.

**8.** Describe briefly the phenomena of diffusion in liquids and gases. Explain clearly why stirring hastens the process of diffusion.

**9.** A vertical tube 50 cm. long contains carbon dioxide in the lower 5 cm. and the remainder of the tube contains air. Diffusion is found to be completed in 1 hour 20 minutes. Supposing the proportions of the gases to be the same, in what time would diffusion be completed in a tube 10 cm. long?

**10.** Give a brief description of the phenomenon of osmosis. Describe an experiment for illustrating osmosis.

**11.** Describe Graham's method of analysis. Explain the modern conception of a colloidal solution.

**12.** What is meant by the term osmotic pressure? Describe how it may be found for a given salt solution.

**13.** Describe briefly the methods by which a gas may flow through a porous substance, with reference to the size of the pores.

**14.** Describe an experiment to demonstrate that coal-gas diffuses more rapidly than air.

**15.** Describe the most accurate method you know of measuring the surface tension of water.

Water from a depth of 4 cm. drips into the carbide chamber of a bicycle lamp through a nozzle 0·5 mm. in diameter. Show that the lamp can produce intermittently a gas pressure equal to a 10 cm. head of water without blowing back. The surface tension of water is 75 dynes per cm.

C.W.B., H.C.

**16.** What is meant by osmosis? State the laws of osmotic pressure. A 1 per cent. sugar solution at 27° C. has an osmotic pressure of 54 cm. of mercury. Show that this is the pressure which would be exerted by the sugar if its molecules existed as a gas occupying the same volume. (Density of hydrogen at N.T.P. = 0·00009 gm. per c.c. Molecular weight of hydrogen = 2, of sugar = 342.) C.W.B., H.C.

**17.** Describe some experiments illustrating the phenomena of surface tension.

In a measurement of the surface tension of a liquid by the capillary tube method, the height the liquid rises in a tube of 1 mm. radius is 1·5 cm. If the density of the liquid is 0·8 gm. per c.c. and the angle of contact zero, what is the surface tension of the liquid? Prove any formula you use.

L.U.Hi.Sch.

**18.** Using glass capillary tubing, a reading microscope and the ordinary apparatus of a physics laboratory, how would you make an accurate determination of the surface tension between glass and water?

Two plane glass plates, in contact along a vertical line and inclined to each other, are partly immersed in a liquid of density 1·06 gm. per c.c. and

surface tension 52 dynes per cm. If the level of the liquid between the plates at 1 cm. from the line of contact is different from that outside by 1 cm., what is the angle between the plates? (Assume $g = 980$ cm./sec.$^2$ and contact angle $= 0°$.) J.M.B., H.S.C.

**19.** A spherical bubble of radius 2 cm. is blown in an atmosphere whose pressure is $10^6$ dynes per sq. cm. If the surface tension of the liquid composing the film is 60 dynes per cm., to what pressure must the surrounding atmosphere be brought in order exactly to double the radius of the bubble? Assume no temperature change and no diffusion through the bubble.
J.M.B., H.S.C.

**20.** Describe how you would measure the variation with temperature of the surface tension of a liquid.

A capillary U tube contains a liquid of surface tension 75 dyne cm.$^{-1}$. One limb of the tube has internal diameter 2 mm. and the other 0·4 mm.. and the tube contains a liquid of density 1·2 gm. cm.$^{-3}$ which wets the glass. What is the difference of level of meniscus in the two limbs? L.U.

# LOGARITHMIC TABLES

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **10** | 0000 | 0043 | 0086 | 0128 | 0170 | | | | | | 4 | 9 | 13 | 17 | 21 | 26 | 30 | 34 | 38 |
| | | | | | | 0212 | 0253 | 0294 | 0334 | 0374 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 |
| **11** | 0414 | 0453 | 0492 | 0531 | 0569 | | | | | | 4 | 8 | 12 | 15 | 19 | 23 | 27 | 31 | 35 |
| | | | | | | 0607 | 0645 | 0682 | 0719 | 0755 | 4 | 7 | 11 | 15 | 19 | 22 | 26 | 30 | 33 |
| **12** | 0792 | 0828 | 0864 | 0899 | 0934 | | | | | | 3 | 7 | 11 | 14 | 18 | 21 | 25 | 28 | 32 |
| | | | | | | 0969 | 1004 | 1038 | 1072 | 1106 | 3 | 7 | 10 | 14 | 17 | 20 | 24 | 27 | 31 |
| **13** | 1139 | 1173 | 1206 | 1239 | 1271 | | | | | | 3 | 7 | 10 | 13 | 16 | 20 | 23 | 26 | 30 |
| | | | | | | 1303 | 1335 | 1367 | 1399 | 1430 | 3 | 7 | 10 | 13 | 16 | 19 | 22 | 25 | 29 |
| **14** | 1461 | 1492 | 1523 | 1553 | 1584 | | | | | | 3 | 6 | 9 | 12 | 15 | 19 | 22 | 25 | 28 |
| | | | | | | 1614 | 1644 | 1673 | 1703 | 1732 | 3 | 6 | 9 | 12 | 15 | 17 | 20 | 23 | 26 |
| **15** | 1761 | 1790 | 1818 | 1847 | 1875 | | | | | | 3 | 6 | 9 | 11 | 14 | 17 | 20 | 23 | 26 |
| | | | | | | 1903 | 1931 | 1959 | 1987 | 2014 | 3 | 6 | 8 | 11 | 14 | 17 | 19 | 22 | 25 |
| **16** | 2041 | 2068 | 2095 | 2122 | 2148 | | | | | | 3 | 5 | 8 | 11 | 14 | 16 | 19 | 22 | 24 |
| | | | | | | 2175 | 2201 | 2227 | 2253 | 2279 | 3 | 5 | 8 | 10 | 13 | 16 | 18 | 21 | 23 |
| **17** | 2304 | 2330 | 2355 | 2380 | 2405 | | | | | | 3 | 5 | 8 | 10 | 13 | 15 | 18 | 20 | 23 |
| | | | | | | 2430 | 2455 | 2480 | 2504 | 2529 | 2 | 5 | 7 | 10 | 12 | 15 | 17 | 20 | 22 |
| **18** | 2553 | 2577 | 2601 | 2625 | 2648 | | | | | | 2 | 5 | 7 | 9 | 12 | 14 | 16 | 19 | 21 |
| | | | | | | 2672 | 2695 | 2718 | 2742 | 2765 | 2 | 5 | 7 | 9 | 11 | 14 | 16 | 18 | 21 |
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| **20** | 3010 | 3032 | 3054 | 3075 | 3096 | 3118 | 3139 | 3160 | 3181 | 3201 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| **21** | 3222 | 3243 | 3263 | 3284 | 3304 | 3324 | 3345 | 3365 | 3385 | 3404 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| **22** | 3424 | 3444 | 3464 | 3483 | 3502 | 3522 | 3541 | 3560 | 3579 | 3598 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| **23** | 3617 | 3636 | 3655 | 3674 | 3692 | 3711 | 3729 | 3747 | 3766 | 3784 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
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| **25** | 3979 | 3997 | 4014 | 4031 | 4048 | 4065 | 4082 | 4099 | 4116 | 4133 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 15 |
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| **28** | 4472 | 4487 | 4502 | 4518 | 4533 | 4548 | 4564 | 4579 | 4594 | 4609 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| **29** | 4624 | 4639 | 4654 | 4669 | 4683 | 4698 | 4713 | 4728 | 4742 | 4757 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
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| **37** | 5682 | 5694 | 5705 | 5717 | 5729 | 5740 | 5752 | 5763 | 5775 | 5786 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
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| **42** | 6232 | 6243 | 6253 | 6263 | 6274 | 6284 | 6294 | 6304 | 6314 | 6325 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
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| **46** | 6628 | 6637 | 6646 | 6656 | 6665 | 6675 | 6684 | 6693 | 6702 | 6712 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7 | 8 |
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| **49** | 6902 | 6911 | 6920 | 6928 | 6937 | 6946 | 6955 | 6964 | 6972 | 6981 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
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| 56 | 7482 | 7490 | 7497 | 7505 | 7513 | 7520 | 7528 | 7536 | 7543 | 7551 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 57 | 7559 | 7566 | 7574 | 7582 | 7589 | 7597 | 7604 | 7612 | 7619 | 7627 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 58 | 7634 | 7642 | 7649 | 7657 | 7664 | 7672 | 7679 | 7686 | 7694 | 7701 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
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| 60 | 7782 | 7789 | 7796 | 7803 | 7810 | 7818 | 7825 | 7832 | 7839 | 7846 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 61 | 7853 | 7860 | 7868 | 7875 | 7882 | 7889 | 7896 | 7903 | 7910 | 7917 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 62 | 7924 | 7931 | 7938 | 7945 | 7952 | 7959 | 7966 | 7973 | 7980 | 7987 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 6 |
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| 70 | 8451 | 8457 | 8463 | 8470 | 8476 | 8482 | 8488 | 8494 | 8500 | 8506 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
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| 72 | 8573 | 8579 | 8585 | 8591 | 8597 | 8603 | 8609 | 8615 | 8621 | 8627 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
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| 78 | 8921 | 8927 | 8932 | 8938 | 8943 | 8949 | 8954 | 8960 | 8965 | 8971 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 79 | 8976 | 8982 | 8987 | 8993 | 8998 | 9004 | 9009 | 9015 | 9020 | 9025 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
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| 82 | 9138 | 9143 | 9149 | 9154 | 9159 | 9165 | 9170 | 9175 | 9180 | 9186 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
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| ·49 | 3090 | 3097 | 3105 | 3112 | 3119 | 3126 | 3133 | 3141 | 3148 | 3155 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 50 | 3162 | 3170 | 3177 | 3184 | 3192 | 3199 | 3206 | 3214 | 3221 | 3228 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| ·51 | 3236 | 3243 | 3251 | 3258 | 3266 | 3273 | 3281 | 3289 | 3296 | 3304 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| ·52 | 3311 | 3319 | 3327 | 3334 | 3342 | 3350 | 3357 | 3365 | 3373 | 3381 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| ·53 | 3388 | 3396 | 3404 | 3412 | 3420 | 3428 | 3436 | 3443 | 3451 | 3459 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| ·54 | 3467 | 3475 | 3483 | 3491 | 3499 | 3508 | 3516 | 3524 | 3532 | 3540 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| ·55 | 3548 | 3556 | 3565 | 3573 | 3581 | 3589 | 3597 | 3606 | 3614 | 3622 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| ·56 | 3631 | 3639 | 3648 | 3656 | 3664 | 3673 | 3681 | 3690 | 3698 | 3707 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| ·57 | 3715 | 3724 | 3733 | 3741 | 3750 | 3758 | 3767 | 3776 | 3784 | 3793 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| ·58 | 3802 | 3811 | 3819 | 3828 | 3837 | 3846 | 3855 | 3864 | 3873 | 3882 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| ·59 | 3890 | 3899 | 3908 | 3917 | 3926 | 3936 | 3945 | 3954 | 3963 | 3972 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| ·60 | 3981 | 3990 | 3999 | 4009 | 4018 | 4027 | 4036 | 4046 | 4055 | 4064 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 7 | 8 |
| ·61 | 4074 | 4083 | 4093 | 4102 | 4111 | 4121 | 4130 | 4140 | 4150 | 4159 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·62 | 4169 | 4178 | 4188 | 4198 | 4207 | 4217 | 4227 | 4236 | 4246 | 4256 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·63 | 4266 | 4276 | 4285 | 4295 | 4305 | 4315 | 4325 | 4335 | 4345 | 4355 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·64 | 4365 | 4375 | 4385 | 4395 | 4406 | 4416 | 4426 | 4436 | 4446 | 4457 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·65 | 4467 | 4477 | 4487 | 4498 | 4508 | 4519 | 4529 | 4539 | 4550 | 4560 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·66 | 4571 | 4581 | 4592 | 4603 | 4613 | 4624 | 4634 | 4645 | 4656 | 4667 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 10 |
| 67 | 4677 | 4688 | 4699 | 4710 | 4721 | 4732 | 4742 | 4753 | 4764 | 4775 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| ·68 | 4786 | 4797 | 4808 | 4819 | 4831 | 4842 | 4853 | 4864 | 4875 | 4887 | 1 | 2 | 3 | 4 | 6 | 7 | 8 | 9 | 10 |
| ·69 | 4898 | 4909 | 4920 | 4932 | 4943 | 4955 | 4966 | 4977 | 4989 | 5000 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| ·70 | 5012 | 5023 | 5035 | 5047 | 5058 | 5070 | 5082 | 5093 | 5105 | 5117 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 9 | 11 |
| ·71 | 5129 | 5140 | 5152 | 5164 | 5176 | 5188 | 5200 | 5212 | 5224 | 5236 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| 72 | 5248 | 5260 | 5272 | 5284 | 5297 | 5309 | 5321 | 5333 | 5346 | 5358 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| ·73 | 5370 | 5383 | 5395 | 5408 | 5420 | 5433 | 5445 | 5458 | 5470 | 5483 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| ·74 | 5495 | 5508 | 5521 | 5534 | 5546 | 5559 | 5572 | 5585 | 5598 | 5610 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| 75 | 5623 | 5636 | 5649 | 5662 | 5675 | 5689 | 5702 | 5715 | 5728 | 5741 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 10 | 12 |
| ·76 | 5754 | 5768 | 5781 | 5794 | 5808 | 5821 | 5834 | 5848 | 5861 | 5875 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| ·77 | 5888 | 5902 | 5916 | 5929 | 5943 | 5957 | 5970 | 5984 | 5998 | 6012 | 1 | 3 | 4 | 5 | 7 | 8 | 10 | 11 | 12 |
| ·78 | 6026 | 6039 | 6053 | 6067 | 6081 | 6095 | 6109 | 6124 | 6138 | 6152 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 13 |
| ·79 | 6166 | 6180 | 6194 | 6209 | 6223 | 6237 | 6252 | 6266 | 6281 | 6295 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| 80 | 6310 | 6324 | 6339 | 6353 | 6368 | 6383 | 6397 | 6412 | 6427 | 6442 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| ·81 | 6457 | 6471 | 6486 | 6501 | 6516 | 6531 | 6546 | 6561 | 6577 | 6592 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| 82 | 6607 | 6622 | 6637 | 6653 | 6668 | 6683 | 6699 | 6714 | 6730 | 6745 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| ·83 | 6761 | 6776 | 6792 | 6808 | 6823 | 6839 | 6855 | 6871 | 6887 | 6902 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| ·84 | 6918 | 6934 | 6950 | 6966 | 6982 | 6998 | 7015 | 7031 | 7047 | 7063 | 2 | 3 | 5 | 6 | 8 | 10 | 11 | 13 | 15 |
| 85 | 7079 | 7096 | 7112 | 7129 | 7145 | 7161 | 7178 | 7194 | 7211 | 7228 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| ·86 | 7244 | 7261 | 7278 | 7295 | 7311 | 7328 | 7345 | 7362 | 7379 | 7396 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| ·87 | 7413 | 7430 | 7447 | 7464 | 7482 | 7499 | 7516 | 7534 | 7551 | 7568 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 16 |
| 88 | 7586 | 7603 | 7621 | 7638 | 7656 | 7674 | 7691 | 7709 | 7727 | 7745 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| ·89 | 7762 | 7780 | 7798 | 7816 | 7834 | 7852 | 7870 | 7889 | 7907 | 7925 | 2 | 4 | 5 | 7 | 9 | 11 | 13 | 14 | 16 |
| 90 | 7943 | 7962 | 7980 | 7998 | 8017 | 8035 | 8054 | 8072 | 8091 | 8110 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
| ·91 | 8128 | 8147 | 8166 | 8185 | 8204 | 8222 | 8241 | 8260 | 8279 | 8299 | 2 | 4 | 6 | 8 | 9 | 11 | 13 | 15 | 17 |
| ·92 | 8318 | 8337 | 8356 | 8375 | 8395 | 8414 | 8433 | 8453 | 8472 | 8492 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| ·93 | 8511 | 8531 | 8551 | 8570 | 8590 | 8610 | 8630 | 8650 | 8670 | 8690 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| ·94 | 8710 | 8730 | 8750 | 8770 | 8790 | 8810 | 8831 | 8851 | 8872 | 8892 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| ·95 | 8913 | 8933 | 8954 | 8974 | 8995 | 9016 | 9036 | 9057 | 9078 | 9099 | 2 | 4 | 6 | 8 | 10 | 12 | 15 | 17 | 19 |
| ·96 | 9120 | 9141 | 9162 | 9183 | 9204 | 9226 | 9247 | 9268 | 9290 | 9311 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| ·97 | 9333 | 9354 | 9376 | 9397 | 9419 | 9441 | 9462 | 9484 | 9506 | 9528 | 2 | 4 | 7 | 9 | 11 | 13 | 15 | 17 | 20 |
| ·98 | 9550 | 9572 | 9594 | 9616 | 9638 | 9661 | 9683 | 9705 | 9727 | 9750 | 2 | 4 | 7 | 9 | 11 | 13 | 16 | 18 | 20 |
| ·99 | 9772 | 9795 | 9817 | 9840 | 9863 | 9886 | 9908 | 9931 | 9954 | 9977 | 2 | 5 | 7 | 9 | 11 | 14 | 16 | 18 | 20 |

# TRIGONOMETRICAL TABLE.

| Angle. | Radians. | Sine. | Tangent. | Cotangent. | Cosine. | | |
|---|---|---|---|---|---|---|---|
| 0° | 0 | 0 | 0 | ∞ | 1 | 1·5708 | 90° |
| 1 | ·0175 | ·0175 | ·0175 | 57·2900 | ·9998 | 1·5533 | 89 |
| 2 | ·0349 | ·0349 | ·0349 | 28·6363 | ·9994 | 1·5359 | 88 |
| 3 | ·0524 | ·0523 | ·0524 | 19·0811 | ·9986 | 1·5184 | 87 |
| 4 | 0698 | ·0698 | ·0699 | 14·3006 | ·9976 | 1·5010 | 86 |
| 5 | ·0873 | ·0872 | ·0875 | 11·4301 | ·9962 | 1·4835 | 85 |
| 6 | ·1047 | ·1045 | ·1051 | 9·5144 | ·9945 | 1·4661 | 84 |
| 7 | ·1222 | ·1219 | ·1228 | 8·1443 | ·9925 | 1·4486 | 83 |
| 8 | ·1396 | ·1392 | ·1405 | 7·1154 | ·9903 | 1 4312 | 82 |
| 9 | ·1571 | ·1564 | ·1584 | 6·3138 | ·9877 | 1·4137 | 81 |
| 10 | ·1745 | ·1736 | ·1763 | 5·6713 | ·9848 | 1·3963 | 80 |
| 11 | ·1920 | ·1908 | ·1944 | 5·1446 | ·9816 | 1·3788 | 79 |
| 12 | ·2094 | ·2079 | ·2126 | 4·7046 | ·9781 | 1·3614 | 78 |
| 13 | ·2269 | ·2250 | ·2309 | 4·3315 | ·9744 | 1·3439 | 77 |
| 14 | ·2443 | ·2419 | ·2493 | 4·0108 | ·9703 | 1·3265 | 76 |
| 15 | ·2618 | ·2588 | ·2679 | 3·7321 | ·9659 | 1·3090 | 75 |
| 16 | ·2793 | ·2756 | ·2867 | 3·4874 | ·9613 | 1·2915 | 74 |
| 17 | ·2967 | ·2924 | ·3057 | 3·2709 | ·9563 | 1·2741 | 73 |
| 18 | ·3142 | ·3090 | ·3249 | 3·0777 | ·9511 | 1·2566 | 72 |
| 19 | ·3316 | ·3256 | ·3443 | 2·9042 | ·9455 | 1·2392 | 71 |
| 20 | ·3491 | ·3420 | ·3640 | 2·7475 | ·9397 | 1·2217 | 70 |
| 21 | ·3665 | ·3584 | ·3839 | 2·6051 | ·9336 | 1·2043 | 69 |
| 22 | ·3840 | ·3746 | ·4040 | 2·4751 | ·9272 | 1·1868 | 68 |
| 23 | ·4014 | ·3907 | ·4245 | 2·3559 | ·9205 | 1·1694 | 67 |
| 24 | ·4189 | ·4067 | ·4452 | 2·2460 | ·9135 | 1·1519 | 66 |
| 25 | ·4363 | ·4226 | ·4663 | 2·1445 | ·9063 | 1·1345 | 65 |
| 26 | ·4538 | ·4384 | ·4877 | 2 0503 | ·8988 | 1·1170 | 64 |
| 27 | ·4712 | ·4540 | ·5095 | 1·9626 | ·8910 | 1·0996 | 63 |
| 28 | ·4887 | ·4695 | ·5317 | 1·8807 | ·8830 | 1·0821 | 62 |
| 29 | ·5061 | ·4848 | ·5543 | 1·8040 | ·8746 | 1·0647 | 61 |
| 30 | ·5236 | ·5000 | ·5774 | 1·7321 | ·8660 | 1·0472 | 60 |
| 31 | ·5411 | ·5150 | ·6009 | 1·6643 | ·8572 | 1·0297 | 59 |
| 32 | ·5585 | ·5299 | ·6249 | 1·6003 | ·8480 | 1·0123 | 58 |
| 33 | ·5760 | ·5446 | ·6494 | 1 5399 | ·8387 | ·9948 | 57 |
| 34 | ·5934 | ·5592 | ·6745 | 1·4826 | ·8290 | ·9774 | 56 |
| 35 | ·6109 | ·5736 | ·7002 | 1·4281 | ·8192 | ·9599 | 55 |
| 36 | ·6283 | ·5878 | ·7265 | 1·3764 | ·8090 | ·9425 | 54 |
| 37 | ·6458 | ·6018 | ·7536 | 1·3270 | ·7986 | ·9250 | 53 |
| 38 | ·6632 | ·6157 | ·7813 | 1·2799 | ·7880 | ·9076 | 52 |
| 39 | ·6807 | ·6293 | ·8098 | 1·2349 | ·7771 | ·8901 | 51 |
| 40 | ·6981 | ·6428 | ·8391 | 1·1918 | ·7660 | ·8727 | 50 |
| 41 | ·7156 | ·6561 | ·8693 | 1·1504 | ·7547 | ·8552 | 49 |
| 42 | ·7330 | ·6691 | ·9004 | 1·1106 | ·7431 | ·8378 | 48 |
| 43 | ·7505 | ·6820 | ·9325 | 1·0724 | ·7314 | ·8203 | 47 |
| 44 | ·7679 | ·6947 | ·9657 | 1·0355 | ·7193 | ·8029 | 46 |
| 45 | ·7854 | ·7071 | 1·0000 | 1·0000 | ·7071 | ·7854 | 45 |
| | | Cosine. | Cotangent. | Tangent. | Sine. | Radians. | Angle. |

# ANSWERS

## PART I. DYNAMICS

### Chapter I. p. 6

**1.** Miles × 1·609 = kilometres ; 5·129 kilometres.
**2.** 9 ft. 7·75 in. **3.** 154 sq. cm. ; 1·54 grams wt.
**4.** 381·9 cub. in. ; 99·45 pounds. **5.** 19,500 lb. wt. **6.** 17·49 lb. wt
**7.** 3·055 lb. wt. **8.** 8·710 inches. **9.** 1·125 : 0·96 : 1. **10.** 2·347.
**13.** $48 \times 10^{23}$ kgm. ; 5·304.

### Chapter II. p. 17

**1.** Length of forward reading vernier 1·2 inches ; vernier has 25 divisions.
Length of backward reading vernier 1·3 inches ; vernier has 25 divisions.
**2.** Length of vernier, 59 circle divisions ; vernier has 30 divisions.
**3.** 20 divisions on thimble scale. **4.** 250 divisions. **5.** 5·013 cm.
**6.** 81·05 cm. **7.** 0·0660 mm. **8.** 0·288 lb. per cub. inch.
**11.** 2·6356 mm. **12.** 24·467 mm. ; 7668 cub. mm.

### Chapter III. p. 30

**3.** 49·94 cm. at 19° 53′ east of north. **4.** 3·101 inches at 64° 8′ to OX.
**5.** 12·91 ft./sec. **6.** 0·729 mile. **7.** $\frac{1}{3}$ mile. **8.** 26·67 miles/hour.
**9.** 32·73 miles/hour. **10.** 0·9778 feet/sec.$^2$.
**11.** − 0·02778 metre/sec.$^2$. **12.** − 106·7 feet/sec.$^2$.
**13.** 0·447 metre/sec.$^2$. **14.** 18·75 miles/hour ; 0·0651 mile.
**15.** 0·2291 mile. **16.** 367·1 seconds ; 415·1 seconds.
**17.** − 1·613 feet/sec.$^2$. **18.** 31·32 metres/sec. ; 3·19 seconds.
**19.** 98·28 feet/sec. ; 6·104 seconds. **20.** 100·6 feet.
**21.** 101·2 feet. **22.** 116·6 feet/sec. ; 204·9 feet.
**23.** 400 feet ; 181·3 feet/sec. (taking $g = 32$ feet/sec.$^2$).
**24.** 1 second ; 48 feet above the ground.
**25.**

| No. of body | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Distance from top, in feet | 196 | 144 | 100 | 64 | 36 | 16 | 4 | 0 |

Relative velocity = 16 feet/sec., downwards.
**26.** Acceleration is not uniform.
**27.** 139 feet ; 94·32 feet/sec. ; 4·947 seconds. **29.** $\frac{1}{3}$ mile.
**30.**

| Accel., ft./sec.$^2$ | 80 | 70 | 60 | 35 | 0 | − 10 | − 35 | − 60 | − 70 | − 80 |
|---|---|---|---|---|---|---|---|---|---|---|
| Displacement, ft. | 1·6 | 4·6 | 7·2 | 9·1 | 9·8 | 9·6 | 8·7 | 6·8 | 4·2 | 1·4 |

Total displacement = 63 ft.
**31.** 988·3 cm./sec.$^2$. **33.** 1·92 sec. **34.** (*a*) 40·1 ; 32·1 ; (*b*) 4·3 min.

## Chapter IV. p. 42

**1.** 17·32 cm./sec. at 30°; 10 cm./sec. at 60°; zero; 14·14 cm./sec. at 315°.
**2.** 5·563 feet/sec.; 2·248 feet/sec.; 53·93 seconds.
**3.** 1612 feet/sec. at 7° 8′ to the horizontal.
**4.** 18·05 feet/sec. at 57° 48′ east of north.
**5.** 24·18 feet/sec. at 6° 58′ to the vertical.
**6.** (*a*) 65° 33′ to the vertical; (*b*) 62° 51′ and (*c*) 67° 48′ to the vertical.
**7.** 4·106 miles/hour at 76° 56′ with the direction of the rails.
**8.** 36° 52′ with the edge of the platform; 5·999 feet/sec.
**9.** 44·73 feet/sec. from a point 10° 18′ east of south.
**10.** 25 miles/hour at 36° 52′ east of north.
**11.** 8·544 miles/hour at 20° 33′ east of north; 20° 33′ west of south.
**12.** 154·4 miles/hour; 55° 46′ west of north.
**13.** 60 miles/hour at 120° to $Ox$; 1·732 miles.
**14.** 40° 33′ east of north; 35° 38′ east of south.
**15.** (*a*) Relative velocity, 16·22 knots; direction of motion, 17° 6′ east of north; (*b*) Relative velocity, 4·988 knots; direction of motion 72° 54′ east of north.
**17.** 10·62 metres/sec. at 86° 58′ west of north; 0·531 metres/sec.².
**18.** 7·727 feet/sec. in a direction bisecting the angle included between the straight portions of the pipe.
**19.** 4·77 feet/sec. at 26° 59′ to the original direction of motion.
**20.** 720 cm./sec.². **21.** 6·453 feet/sec.².
**22.** 64·49 feet/sec. at 82° 53′ to the horizontal.
**23.**

| Time, sec. | 0 | 0·1 | 0·2 | 0·3 | 0·4 | 0·5 |
|---|---|---|---|---|---|---|
| $x$, feet | 0 | 0·8 | 1·6 | 2·4 | 3·2 | 4·0 |
| $y$, feet | 0 | 0·16 | 0·64 | 1·44 | 2·56 | 4·0 |

**24.** 1500 feet; 1° 54′.
**25.**

| Angle, degrees | 30 | 40 | 45 | 50 | 60 |
|---|---|---|---|---|---|
| Horiz. range, ft. | 130,982 | 148,951 | 151,250 | 148,951 | 130,982 |
| Time, seconds | 68·75 | 88·38 | 97·23 | 105·3 | 119·1 |
| Greatest height, ft. | 18,906 | 31,248 | 37,812 | 44,269 | 56,711 |

**26.** 26° 48′. **27.** Angle of elevation, 4° 29′, or 78° 57′. **30.** 45° 02′.

## Chapter V. p. 56

**1.** 9·425 radians/sec. **2.** 0·1047 radian/sec. **3.** 286·4 revs./min.
**4.** 9·546 revs./min. **5.** 12 radians/sec. **6.** 0·873 radians/sec.².
**7.** −4 radians/sec.². **8.** −1 radian/sec.². **9.** 78·57 seconds; 98·21 revs.
**10.** 750 radians. **11.** 0·0633 radian/sec.². **12.** 15·09 radians/sec.; 720 revs.
**13.** 32 inches. **14.** 60 revs./min. **15.** 71·55; 281·7.
**16.** 40·74 revs./min. **17.** 1 : 8. **18.** 24 teeth. **19.** $\dfrac{v \times ON}{OP^2}$

## Chapter VI. p. 65

**1.** 675 poundals. **2.** 3·816 cm./sec.².
**3.** Dynes × 0·000072328 = poundals; poundals × 13,826 = dynes.

4. 34·06 poundals.
5. 4·141 tons wt.
6. (a) 1000 lb. wt.; (b) 937·9 lb. wt.; (c) 1062·1 lb. wt.
7. 2·236 tons wt.
8. 160 lb. wt.; 80 lb. wt.; he would have a downward acceleration of 0·5 $g$.
9. 2·927 feet/sec.; 29·27 poundals; that the table offers a resistance to sliding equal to 1 lb. wt.; 1 lb. wt.
10. 280·3 cm./sec.$^2$; 630,600 dynes.
11. Close agreement; actual $a$ : theoretical $a = 2·785 : 2·8$.
12. 4·5 miles; 28·7 lb. wt.
13. 264 ton-feet/sec.; 0·4099 ton wt.
14. 1,800,000 pound-feet/sec.; 5,590,000 lb. wt.
16. 0·976 feet/sec.; arithmetical sum of momenta = 2·013 pound-feet/sec.
17. 23·06 pound-feet/sec.; at 116° 34′ to initial direction of motion.
18. 30 ounce-feet/sec.; 5 feet/sec.; 1·035 feet.
19. 40° 54′; 33·58 pound-feet/sec.
20. (a) 2·236 seconds; (b) 3·354 seconds (taking $g = 32$ feet/sec.$^2$).
22. 40 feet/sec.; 30°.
23. 6 ft.; 10 ft.; 2·73 sec.
25. 5° 19′.
26. Force $= (M + m)g \tan \alpha$.
27. 32·64 ft./sec.

## Chapter VII. p. 81

1. 10·58 lb. wt. at 19° 8′ to the 8 lb. force.
2. 6·928 lb. wt. at 30° to the 8 lb. force.
3. 1·951 kg. wt. at 41° 13′ to the resultant.
4. 27° 40′ between 7 lb. and 10 lb.; 40° 32′ between 5 lb. and 10 lb.
5. 5·292 lb. wt.; 48° 36′.
6. 3·085 lb. wt.; 40° 30′.
7.

| Angle, degrees | 165 | 170 | 174 | 178 | 180 |
|---|---|---|---|---|---|
| Equilibrant, lb. wt. | 2·610 | 1·744 | 1·046 | 0·3500 | 0 |

8. (a) $P = 1·2856$ lb. wt.; $R = 1·532$ lb. wt.; (b) $P = 1·6782$ lb. wt.; $R = 2·6108$ lb. wt.; (c) $P = 1·368$ lb. wt.; $R = 1·064$ lb. wt.; (d) $P = 1·485$ lb. wt.; $R = 2·274$ lb. wt.
9. 13·61$m$ poundals; 13·61 feet/sec.$^2$; 1·084 seconds.
10. 5·571 lb. wt.; 3·571 lb. wt.
11. 2·996 lb. wt. at 26° 27′ to the vertical.
12. 102° 38′; 120°.
13. Coordinates of the weight are 3·2 and −2·4 feet; 41·67 lb. wt.
14. 89·94 lb. wt.; 113·7 lb. wt. at 34° 36′ to AC.
15. 0·16 ton wt. in AB; 0·89 ton wt. in AC.
16. 5·28 tons wt.; 2·36 tons wt.
17. 7·78 tons wt.; 4·86 tons wt.
18. 19·24 tons wt.; 7·76 tons wt.
19. 1 lb. wt. in direction from O to D
20. $P = 28·28$, $S = 45·95$, $V = 17·67$, all in tons wt.
21. 1·732 tons wt.; 11·928 tons wt.
22. $P = W \sin \theta$; $P \cos \alpha + Q \cos \beta + R \cos \gamma = W \sin \theta$; $P \sin \alpha + Q \sin \beta + R \sin \gamma = 0$.
24.

| Distance of knot from A, feet | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Tension, lb. wt. | 4 | 4 | 3·85 | 3·65 | 3·35 | 3·05 | 2·5 | 1·85 | 0·67 | 0 | 0 |

25. 16,000 lb. wt.; 8000 lb. wt.
26. 6·25 lb. wt.; 5 lb. wt.

27. $AD = \dfrac{\sqrt{m^2 . AB^2 + n^2 . AC^2 + 2mn . AB . AC . \cos BAC}}{m+n}$

$DAC = \tan^{-1} \dfrac{m . AB}{m . AB \cos BAC + n . AC}$.

28. $W \sin\beta/\sin(\alpha+\beta)$ ; $W \sin\alpha/\sin(\alpha+\beta)$ ; $W \sin\beta/\sin(\alpha+\beta)$ ; $2W \sin\alpha/\sin(\alpha+\beta)$.

29. 4·954 ft./sec.$^2$ down plane ; 193·1 poundals at 30° to vertical.

## Chapter VIII. p. 92

1.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Angle, degrees | 0 | 30 | 60 | 90 | 120 | 150 | 180 |
| Turning moment, lb.-inches | 0 | 70 | 121·2 | 140 | 121·2 | 70 | 0 |

2. Two positions differing by 180° ; OA makes 26° 34′ with the vertical through O.
3. 14 lb. wt., falling between the given forces at 5·143 inches from the 8 lb. wt.
4. 2 lb. wt., falling outside the given forces ; of same sense as, and distant 36 inches from the 8 lb. wt.
5. At 0·667 foot from the pivot, on the side opposite to the 12 lb. wt.
6. 10·95 inches from A.
7. 1·667 tons wt. ; 0·833 ton wt.
8. 1425 lb. wt. ; 3150 lb. wt.
9. 20 kilograms at 59·5 cm. from A.
10. 2·309 lb. wt. at 30° to the vertical ; 1·527 lb. wt. at 49° 6′ to the vertical.
11. 2·506 lb. wt., vertical ; 1·856 lb. wt. at 47° 29′ to the vertical.
12. 23·53 lb. wt. ; 423·53 lb. wt.
13. Reaction at A = 220·6 lb. wt. at 24° 56′ to the vertical ; reaction at B = 93 lb. wt., horizontal.
14. 96·22 lb. wt. ; 178·2 lb. wt. at 32° 41′ to the vertical.
15. Reaction at A = 2·267 tons wt. ; reaction at B = 25·73 tons wt.

16.

| | | | | | |
|---|---|---|---|---|---|
| Distance from left-hand support, feet | 2 | 4 | 6 | 8 | 10 |
| Reaction of left-hand support, lb. wt. | 125 | 100 | 75 | 50 | 25 |
| Reaction of right-hand support, lb. wt. | 25 | 50 | 75 | 100 | 125 |

17.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Distance of A, feet | 12 | 10 | 8 | 6 | 4 | 2 | 0 |
| Reaction, lb. wt. | 0 | 25 | 50 | 100 | 150 | 200 | 250 |

18. 742 lb. wt., at 5·622 feet from bow.
19. Both spring balances are attached at points on the rod between the loads ; one is at 3 inches from the 3 lb. wt., and the other is at 7 inches from the 2 lb. wt.
20. 60°.
21. 15 lb. wt., parallel to CA.

## Chapter IX. p. 109

1. 4750 lb. wt. ; 1750 lb. wt.
2. 400 lb. wt.
3. 19·2 feet from A.
4. 16·67 lb. wt.
5. $\bar{x} = 3·083$ inches ; $\bar{y} = 4·583$ inches.
6. G is 2·37 inches from A and 2·65 inches from B ; 0·81 lb. wt.
7. In the median, 4·33 inches from the 18 inches side.
8. Coordinates of G from centre of plate, 0·72 and 0·37 inches.
9. $\bar{x} = 3·32$ inches , $\bar{y} = 4·51$ inches ; $\bar{z} = 6·49$ inches.

**10.** 133·3 and 266·7 lb. wt.; 400 lb. wt.; 0 and 800 lb. wt.
**11.** 600 lb. wt. **12.** 26° 34′.
**13.** Taking the origin at D, $\bar{x}=5·59$ feet; $\bar{y}=6·54$ feet; P = 654,000 lb. wt.
**15.** 5·825 inches. **16.** AB makes 65° with the vertical.
**17.** 1·72 inches from AB. **18.** 0·55949 lb. wt.; 0·00001 lb. wt. too much.
**19.** 237·1 grams wt. **21.** 51 degrees.
**22.** In the radius bisecting the quadrant, at $\frac{4\sqrt{2}}{3} \cdot \frac{r}{\pi}$ from the centre of the circle.
**23.** The zero mark on FDE is $\left(\frac{W_0c - W_1d}{w}\right)$ from F; the length of graduation corresponding to unit load in the scale pan is $c/w$; $c = 0·25$ inch; $W_1 = 0·25(W_0 - 1)$.
**24.** $\frac{2}{9}s$ and $\frac{7}{18}s$, where $s$ = the side of the square.
**25.** 22° 37′. **26.** 83·6 inches. **29.** In axis; 0·399$a$. **30.** 0·0554°; No.

## Chapter X. p. 123

1. Two opposing couples; a force of 400 lb. wt. along each long edge; a force of 133·3 lb. wt. along each short edge.
2. Top hinge, upward pull of 75 lb. wt. away from the door at 36° 52′ to the vertical; bottom hinge, upward push of 75 lb. wt. towards the door at 36° 52′ to the vertical.
3. A vertical force of 5 tons wt. in the axis, and a couple of 40 ton inches.
4. 112,000 lb. wt. acting vertically at the centre of the base, and a couple of 186,700 lb.-feet.
5. 20 lb. wt. at B, at 30° to AB produced.
6. The system reduces to a couple, having a moment represented by 2△ABC.
7. R = 2·828 lb. wt., at 45° to the sides of the square, and acting at a point 2 feet from CD produced and 3 feet from AD produced.
8. 36·96 lb. wt. at A, at 23° 6′ to the vertical; 14·5 lb. wt. at B, horizontal.
9. 825·3 lb. wt. at B; 950·2 lb. wt. at A, at 62° 10′ to the horizontal.

**10.**

| $\theta$, degrees | 45 | 30 | 15 | 5 |
|---|---|---|---|---|
| P, lb. wt. | 12·5 | 21·65 | 46·65 | 142·9 |
| S, lb. wt. | 2·5 | 2·5 | 2·5 | 2·5 |
| Q, lb. wt. | 17·68 | 25·00 | 48·30 | 143·4 |

**11.**

| $\theta$ degrees | (a) | | | (b) | | |
|---|---|---|---|---|---|---|
| | P, lb. wt. | S, lb. wt. | Q, lb. wt. | P, lb. wt. | S, lb. wt. | Q, lb. wt. |
| 45 | 10·00 | 0 | 14·14 | 8·75 | −1·25 | 12·37 |
| 30 | 17·32 | 0 | 20·00 | 15·16 | −1·25 | 17·50 |
| 15 | 37·32 | 0 | 38·64 | 32·66 | −1·25 | 33·81 |
| 5 | 114·3 | 0 | 114·7 | 100·0 | −1·25 | 100·4 |

**12.** 184·8 lb. wt. at 39° 40′ to the horizontal.

**13.**

| $x$, feet | 4 | 8 | 12 | 16 | 19 |
|---|---|---|---|---|---|
| P, lb. wt. | 35·80 | 53·12 | 70·44 | 87·76 | 100·75 |

The graph is a straight line.

**14.** 33·33 lb. wt.

**15.** 2·828 lb. wt., parallel to CA and passing through a point on CD produced at twice the side of the square from D.

**16.** 3·749 feet from the end having the rope inclined at 60°.

**17.** Reaction $= \frac{1}{2}$W, horizontal. **19.** 22·66 lb. wt.

**20.** $lW/2a$ ; $W\sqrt{4a^2 - l^2}/2a$.

**21.** 5 lb. wt. ; 5·176 lb. wt. compression ; 4·226 lb. wt. tension ; 2·887 lb. wt. tension.

**22.** T = 14·43 lb. wt. ; $R_x$ = 14·43 lb. wt. ; $R_y$ = 7·5 lb. wt.

**25.**

| Bar | AB | BC | CD | DE | EF | FA | CF |
|---|---|---|---|---|---|---|---|
| Force, lb. wt. | 11·54 | 23·09 | 23·09 | 11·54 | 23·09 | 23·09 | 23·09 |
| Nature of force | Pull | Pull | Pull | Pull | Pull | Pull | Push |

**27.** 16° 16′ ; 120 lb. wt. ; 160 lb. wt. **28.** 5 lb. wt.

## Chapter XI. p. 139

**1.** 2·8 tons wt. per sq. inch. **2.** 12·5 tons wt.
**3.** 1·784 inches. **4.** 86·62 tons wt. **5.** 1·697 tons wt. per sq. inch.
**6.** 0·000806. **7.** 0·000417. **8.** 0·0556 inch.
**9.** 3600 lb. wt. per sq. inch ; 0·000125 ; 28,800,000 lb. wt. per sq. inch.
**10.** 6·11 tons wt. per sq. inch ; 0·000456 ; 0·046 inch.
**11.** 13,440,000 lb. wt. per sq. inch. **12.** 0·0485 inch.
**13.** 0·0794 cubic inch. **14.** 12,520,000 lb. wt. per sq. inch.
**15.** Bending moments : at middle, 15 ton-feet ; at each 1 ton load, 10 ton-feet. Shearing force 1 ton wt.

**16.**

| Distance of section from wall, ft. | 0 | 2 | 4 | 6 | 8 |
|---|---|---|---|---|---|
| Bending moment, lb.-ft. | 1600 | 900 | 400 | 100 | 0 |
| Shearing force, lb. wt. | 400 | 300 | 200 | 100 | 0 |

**17.** 30,190,000 lb. wt. per sq. inch. **18.** 15,000 lb. ft. ; 18,750 lb. ft.
**19.** $R_A$ = 50 lb. wt. ; $R_B$ = 40 lb. wt.

| Distance from A, ft. | 1 | 3 | 5 | 7 |
|---|---|---|---|---|
| Bending moment, lb.-ft. | 50 | 110 | 80 | 0 |
| Shearing force, lb. wt. | 50 | 10 | −40 | −40 |

**20.** $2{\cdot}26 \times 10^{11}$ dynes/sq. cm. **22.** 0·101 *a*.
**23.** 0·00981 sq. cm.

## Chapter XII. p. 150

**1.** 4,032,000 foot-lb. **2.** 4,752,000 foot-lb. **3.** 697,000 foot-lb.
**4.** 24,550,000 foot-lb. **5.** 18,000,000 foot-lb. **6.** 49·7 foot-lb.
**7.** 265,800 foot-tons ; 100·7 tons wt. **8.** 1·559 tons wt.
**9.** 894·4 lb. wt. **10.** 0·64 H.P. **11.** 60·61 H.P. ; 101 H.P.
**12.** 10·96 H.P. ; 48·1 amperes. **13.** 14·5 H.P. **14.** 1796 lb. wt.
**15.** (Energy wasted in impact = 50 ft.-cwt., see p. 220) ; pile is driven 10 inches.
**17.** 3366 lb. wt. ; 62·4 miles/hour. **18.** (*a*) Times are equal : (*b*) $t_A/t_B$ = 0·5.
**19.** $ml^2/t^2$ ; $ml^2/t^3$ ; 9·66. **20.** 21·21 miles/hour ; 30 sec. ; 10·6 sec.
**22.** 17,562 lb. wt. ; 8781 lb. wt. **23.** 550 radians/sec. ; 13·75 sec.

## Chapter XIII. p. 160

1. $\mu = 0{\cdot}267$ ; $\phi = 14^\circ\ 57'$.
2. 3·28 feet.

3.

| $\theta$, degrees | 0 | 15 | 30 | 45 | 60 | 75 |
|---|---|---|---|---|---|---|
| P, lb. wt. | 0·25W | 0·242W | 0·253W | 0·284W | 0·350W | 0·500W |
| Work, foot-lb. | 0·25W | 0·234W | 0·219W | 0·201W | 0·175W | 0·129W |

4.

| $\theta$, degrees | 0 | 15 | 30 | 45 | 60 | 75 | 90 |
|---|---|---|---|---|---|---|---|
| P, lb. wt. | 0·250W | 0·500W | 0·716W | 0·884W | 0·991W | 1·031W | W |
| Work, ft.-lb. | $\infty$ | 1·932W | 1·432W | 1·250W | 1·145W | 1·067W | W |

5.

| $\theta$, degrees | 0 | 15 | 30 | 45 | 60 | 75 | 90 |
|---|---|---|---|---|---|---|---|
| P, lb. wt. | 0·250W | 0·559W | 0·966W | 1·667W | 3·460W | 59·4W | $\infty$ |
| Work, ft.-lb. | $\infty$ | 2·086W | 1·673W | 1·667W | 1·996W | 15·92W | — |

P = $\infty$ when $\theta = 76^\circ$ nearly.

6. 18·21 feet/sec.$^2$.
7. 643 lb. wt.
8. 1 in 55·8.
9. 36 inch-lb.
10. 5·367 feet/sec.$^2$ ; 268·3 poundals.
12. $w(\frac{1}{2}\tan\theta - \mu)$.
13. 0·4.
14. 17·6 ft.
15. 565·5 cm./sec.$^2$.

## Chapter XIV. p. 172

1. Velocity ratio = 3 ; mechanical advantage = 2·143 ; effect of friction = 60 lb. wt. ; efficiency = 71·43 per cent.
2. 5·625 lb. wt. ; 16 ; 20·83 lb. wt.
3. 1·067 lb. wt. ; 5·714 lb. wt.
4. 20 ; 420 lb. wt. ; 14.
5. 32 ; 27·5 ; 90 lb. wt. ; 85·9 per cent.
6. 48 ; 936 lb. wt. ; 31·2 ; 504 lb. wt.
7. 3140 degrees ; 187,900 inch-lb.
8. 18·75 lb. wt. ; 100 per cent.
9. Neglecting friction, 40 lb. wt. ; taking account of friction, 74·3 lb. wt.
10. $P = \frac{1}{15}W + 7\frac{2}{3}$ ; $18\frac{2}{3}$ lb. wt. ; 29·5 per cent.
12. $P_n = \frac{W}{2^n} + \left(\frac{2^n - 1}{2^n}\right) w$, where $w$ = the weight of each pulley, $n$ = the number of pulleys, and friction is neglected ; 161 lb. wt , 160 lb. wt., assuming that there is no fixed pulley attached to the beam.
13. 377·1 ; 93·33 ; 24·75 per cent.
14. 12·37 per cent.
15. Work done = $W(H + \mu B)$ ; mechanical advantage = $L/2H$.
16. Mechanical advantage, neglecting friction = number of ropes passing from the upper to the lower block ; H.P. = 11·07.
17. 56.

## Chapter XV. p. 191

1. 260·3 lb.-feet.
2. 25,143,000 dyne-cm.
3. 45·96 pound and foot units.
4. (*a*) 0·1778 ; (*b*) 4·8 ; (*c*) 1·2 ; all in pound and foot units.
5. (*a*) 0·364 ; (*b*) 0·182 ; (*c*) 0·546 ; all in pound and foot units.
6. (*a*) 0·5625 ; (*b*) 0·2812 ; (*c*) 1·406 ; (*d*) 1·687 ; (*e*) 0·8437 : all in pound and foot units.

**7.** (*a*) 2; (*b*) 4·5; (*c*) 0·5; (*d*) 1·125; (*e*) 1·625; (*f*) 6·5; all in pound and foot units.

**8.** 859·2 pound and foot units. **9.** 21,270 pound and foot units.

**10.** 23·57; 82·49; both in pound and foot units.

**11.** 6·78 ton-feet. **12.** 12·49 pound and foot units.

**13.** 7071 pound, foot and sec. units; 0·366 lb.-feet.

**14.** 2·514 pound, foot and sec. units; 1042 revs./min.

**15.** 166,000 foot-lb.; 1830 foot-lb.

**16.** (*a*) 8·31 foot-lb.; (*b*) 9·62 foot-lb.; (*c*) 17·93 foot-lb.

**17.** Kinetic energy of translation = 2·325 foot-lb.; kinetic energy of rotation = 1·163 foot-lb.

**18.** 1·872 feet/sec.$^2$; 7·488 radians/sec.$^2$.

**19.** A reaches the bottom first. **20.** $\bar{x} = 6{\cdot}03$ inches; $\bar{y} = 6{\cdot}05$ inches.

**21.** $2\pi^2 n^2 \mathrm{I}$ absolute units; 16·3 pound and foot units.

**22.** $0{\cdot}69\sqrt{g}$ radians/sec. **23.** 18·33 feet.

**26.** 122,500 pound and foot units; 4900 pounds.

**27.** $\mathrm{J}\mathrm{G}a/(\mathrm{I}b^2 + \mathrm{J}a^2)$. **29.** 49,050 ergs; 41·93 radians/sec.$^2$.

## Chapter XVI. p. 212

**1.** 1325 lb. wt. **2.** 3270 lb. wt. **3.** 16·7 poundals.

**4.** 230 lb.-feet. **5.** 1·691 tons wt.; 7·971 tons wt.; 12·029 tons wt.

**6.** 49° 33′. **7.** 70° 21′; 64·2 lb. wt.; 0·357.

**8.** 20·95 feet/sec.; 438·8 feet/sec.$^2$.

**9.** 0·2319 second; 18·45 cm.; 13,550 cm./sec.$^2$.

**10.** 0·2038 foot; 26·8 seconds gain per day.

**11.** 7·211 inches; 0·5882 radian.

**12.**

| Revs./min. | 20 | 40 | 60 | 80 | 100 | 120 |
|---|---|---|---|---|---|---|
| H feet | 7·33 | 1·83 | 0·814 | 0·457 | 0·293 | 0·203 |

**13.** 0·904 second; 96·6 poundals; 64·4 revs./min. **14.** 0·0513 foot.

**15.** 7·196 pounds. **16.** 171 and 189 revs./min. **17.** 5·37 inches.

**18.** 27·3 miles/hour. **19.** 98·29 feet/sec. **20.** $v = \sqrt{ga \tan \alpha}$.

**21.** Velocity $= p\mathrm{A} \sin (pt - \alpha)$; 1·95 feet/sec.; 0·07 second nearly.

**23.** 0·328 second. **24.** 0·452 second. **25.** 2·019 revs./sec.

**27.** 0·15 foot/sec.; 0·268 foot/sec.$^2$; 3·503 seconds.

**28.** Pull in upper cord, 5·817 lb. wt.; in lower cord, 4·404 lb. wt.

**31.** 0·74 poundal-foot. **32.** 2·39 sec.

**33.** 111,910, 111,780, 111,910, 111,870 dyne-cm. per radian.

**36.** Increase = 0·1117 ft./sec.$^2$. **37.** 3·455.

**38.** 314·2 cm./sec.; 0·75. **39.** 8·46 ft.; 4·682 sec.; 0·204.

**40.** $ma[(g/b) - 4\pi^2 n^2]$; compression, zero, or tension according as $4\pi^2 n^2 b$ is $<, =,$ or $> g$.

**41.** 122 ft./sec.; 1943 lb. wt.

**42.** (i) Particle describes complete circle; (ii) Max. $\theta < 90°$, and arrangement behaves as a pendulum.

**43.** (*a*) 66·67 cm.; (*b*) 834·1 cm.; 1 : 3·535.

**44.** 2·719 sec. **45.** 97·02 cm.; 1·98 sec. **46.** 10·94 pound-ft.$^2$.

**48.** $\mathrm{T} \propto \sqrt{\bar{y} + r^2/2\bar{y}}$. **53.** 59·57 cm.

## Chapter XVII. p. 225

**1.** 7·143 feet/sec. ; 22·86 foot-poundals.

**2.** $-1{\cdot}429$ feet/sec. (same sense as B) ; 365·7 foot-poundals.

**3.** $v_A = 9{\cdot}086$ metres/sec. ; $v_B = 11{\cdot}89$ metres/sec. ; $69{\cdot}3 \times 10^6$ ergs.

**4.** $v_A = -2{\cdot}571$ metres/sec. ; $v_B = 11{\cdot}43$ metres/sec. ; $1747 \times 10^6$ ergs.

**5.** (*a*) $v_A = 8{\cdot}571$ metres/sec. ; $v_B = 12{\cdot}57$ metres/sec. ;
(*b*) $v_A = -5{\cdot}143$ metres/sec. ; $v_B = 14{\cdot}86$ metres/sec.

**6.** $-144$ feet/sec. ; $-78$ feet/sec.

**7.** Heights in feet : 5·76, 3·69, 2·36, 1·50 ; energy wasted, 0·664 foot-lb.

**8.** 20·58 feet/sec., at 35° 47′ to the normal to the plane.

**9.**

| Angle, degrees | 0 | 30 | 45 | 60 | 90 |
|---|---|---|---|---|---|
| Forces, lb. wt. | 0 | 5·39 | 7·62 | 9·34 | 10·78 |

**10.** $m/M$ ; 69,320 lb. wt. **13.** 1155 feet/sec. **14.** 12 lb. wt. : 20 foot-lb.

**16.** 80·25 pound-foot-sec. units ; 322 ft.-poundals ; 3·21 ft./sec.

**20.** $-6$ ft./sec. ; $+6$ ft./sec. ; 12 ft.-poundals.

## Chapter XVIII. p. 239

**3.** 333·2 grams wt./sq. cm. **4.** 4693 lb. wt./sq. inch.

**5.** 156 and 260 lb. wt./sq. foot. **7.** 2·544 feet ; 15·01 lb. wt./sq. inch.

**8.** 34 feet. **10.** 1000 lb. wt. : 500 lb. wt. ; 250 lb. wt. **11.** 4978 lb. wt.

**12.** 4563 lb. wt. **13.** 15,000 lb. wt. ; 15,910 lb. wt. ; 9000 lb. wt.

**14.** 1000 lb. wt. ; 37,500 lb. wt. **15.** 6154 grams wt.

**16.** 1 113 lb. wt. ; 1·855 lb. wt.

**17.** AB, 1125 lb. wt. ; BC, 1299 lb. wt. · end, 162·4 lb. wt. ; depth, 1·5 feet.

**18.** 16,500 lb. wt. ; 2·06 feet below the top of the door.

**19.** 1963 lb. wt. at a depth of 10·02 feet.

**20.** 450 lb. wt. ; 495 lb. wt. ; 45 lb. wt. at a depth of 2 feet.

**21.** 22,500 lb. wt. ; 3·27 feet.

**22.** 52,500 lb. wt. ; 22,780 lb. wt. ; 57,220 lb. wt. at 23° 27′ to the vertical 12·23 feet from B.

**23.** 16,875 lb. wt. at 3·8 feet from the bottom.

**24.** $ml/t^2$ ; $m/lt^2$ ; $ml^2/t^2$ ; 14·51 lb. wt./sq. inch.

**25.** 20·83 lb. wt. **26.** 21,600 lb. wt. ; 33,250 lb. wt. **27.** 3633 lb. wt.

**28.** In vertical bisecting lamina ; 5·79 ft. from AB.

**29.** $w\pi h^3 \tan^2\alpha$ ; $\frac{2}{3}w\pi h^3 \tan^2\alpha$.

## Chapter XIX. p. 253

**1.** 356 lb. wt. **2.** 56,340 lb. wt. ; 40,570 lb. wt. ; 27,045 lb. wt.

**3.** 56·25 ; 3142 lb. wt. ; 176,700 lb. wt.

**4.** 2765 foot-lb. **5.** 71·6 gallons per hour.

**6.** (*a*) 100,800 foot-lb. ; (*b*) 1,613 foot-lb. ; 806,500 foot-lb.

**7.** 294,300 ergs. **8.** 926 lb. wt./sq. inch.

**9.** 26,950 lb. wt. , 5544 cubic inches ; 323,400 foot-lb

**10.** 4950 lb. wt. ; 59,400 foot-lb.

**11.** 4·725 horse-power ; 3·071 horse-power.

**12.** 164·6 lb. wt./sq. inch. **13.** 1636 c.c. **14.** 9·67 inches.
**15.** 2567 cubic feet. **16.** 1153 lb. wt. **17.** 42·5 feet.
**18.** 4·036 inches ; 2045 lb. wt.

## Chapter XX. p. 264

**1.** 11,160 tons wt. ; 390,600 cubic feet. **2.** 286·7 sq. feet.
**3.** 8270 lb. wt. **4.** 7·454 lb. wt. **5.** 3·555 lb. wt.
**7.** 8·23. **8.** 0·0289 lb. wt. **9.** 7·68.
**10.** 1166 cubic inches ; 28·12 lb. wt. **11.** 7·00.
**12.** 1·807 lb. wt. **13.** 173·7 tons wt. **14.** 8·59 ; 0·2167.
**15.** 8·76 ; 135·9 cm. **16.** 8·69. **17.** 1·2 cm.
**18.** 23·53 cm. **19.** 0·864 ; 0·8698. **21.** 12·6 gm. ; 9·33 cm.
**23.** 27·2 cubic inches. **25.** 15·17 gm./c.c. ; 2/3. **26.** 0·555 ; 8 kgm.
**27.** 51·2923 gm. **29.** 170·7 lb. wt. ; 0·064 cub. ft.
**30.** Angle with horiz. $= \sin^{-1} h/L\sqrt{s}$.

## Chapter XXI. p. 279

**4.** 85·37 foot-lb. **6.** (*a*) 40 ; 33·9 ; 0·559 ; (*b*) 6 ; 67·9 ; 0·559 ; all in foot-lb.
**7.** 9·33 lb. wt./sq. inch. **9.** 12·5 lb. wt./sq. inch.
**11.** 24·07 feet/sec. ; 23·3 feet/sec. **12.** 0·0262 cubic feet/sec.
**13.** 15·57 feet/sec. ; 0·8 inch ; 0·0543 cubic feet/sec.
**14.** 45,000 lb. wt. ; 17·72 horse-power.
**16.** 146,400 foot-lb. ; 213 horse-power. **19.** 17·12 lb. wt. **20.** $2\pi\sqrt{M/sdg}$.

## Chapter XXII. p. 294

**2.** 73·7 dynes/cm. **3.** 5·958 cm. **5.** 2·35 cm.
**6.** 2·10 mm. **7.** 74·3 dynes/cm. **9.** 3·2 minutes.
**17.** 60·17 dynes/cm. **18.** 5·73°. **19.** 124,955 dynes/sq. cm.
**20.** 5·1 cm.

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